...require ... computers to do tedious calculations for obtaining the solution of simultaneous equations.

3.5. NEWMARK'S METHODS

Newmark's method is a systematic procedure using successive numerial integration in which shear force "Q", bending moment "M", slope or rotation "Q" and deflection "y" are obtained. Unless stated, the following sign convention, which is universally accepted will be adopted in all further discussion.

(i) Shear Force (S.F.) : A structural member which is acted upon by a system of external loads at right angles to its axis is known as beam. The shear force at the cross-section of a beam is defined as the unbalanced vertical force to the right or left of the section. We know that as the shear force is the unbalanced vertical force, there fore it tend to slide one portion of the beam, upwards or downwards with respect to the other. The shear force "Q" which will be taken to be positive is shown in figure (3.4) and figure (3.5) below :

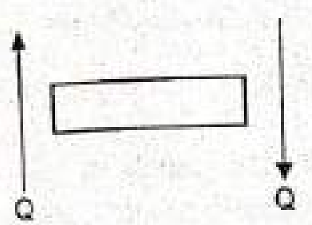

Figure 3.4

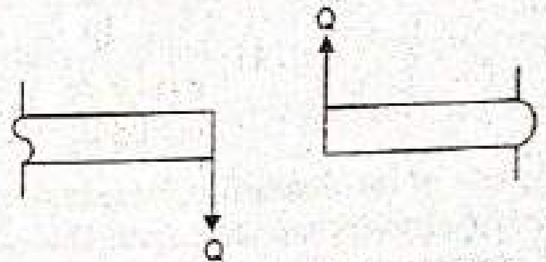

Figure 3.5

We take the shear force at a section as positive, when the left hand portion tends to slide upward or the right hand portion tends to slide downwards or in other words shear force "Q" acting at the ends of an infinitesimally small element show in figure (3.4) produce a clockwise couple. When a particular cross-section of a member is being considered, the positive shear force either side of the section act in the direction as shown in figure (3.5).

Similarly we take shear force at a section as negative, when the left hand portion tends to slide downword or the right hand portion tends to slide upwards. The shear strain and shear stress corresponding to positive shear force will also be taken to be positive.

(ii) Bending Couple and Bending Moment (B.M.) : The bending moment at the cross-section of a beam, is defined as the algebric sum of the moments of the forces, to the right or left of the section. The clockwise and anti-clockwise bending couples are taken to be positive and negative respectively. This sign convention is usually referred to as the frame convention. We take the bending moment at a section as positive if it tends to bend the beam at that point to a curvature having concavity at the top as shown in figure (3.6) below :

Figure 3.6

where as, we take the bending moment at a section as negative if it tends to bend the beam at that point to a curvature having convexity at the top as shown in figure (3.7) below :

Scanned with CamScanner

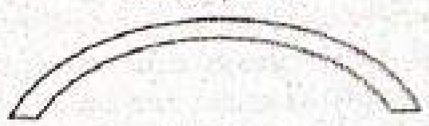

Figure 3.7

Positive bending moment is often called sagging moment where as negative bending moment is often called hogging moment.

Since the bending moment comprises a pair of bending couples, one clockwise and the other anti-clockwise the sign convention for bending couples cannot be carried over to bending moment.

By adopting above sign conventions and taking upward forces to be positive, the relationships between the intensity of load W, shear force Q, bending moment M, curvature ϕ, slope or rotaion θ and deflection y may be expressed as :

$$Q = \int W dx \quad \text{...III}(a)$$

$$M = \int Q\, dx \quad \text{...III}(b)$$

$$\phi = -\frac{M}{EI} \quad \text{...III}(c)$$

$$\theta = \int \phi\, dx \quad \text{...III}(d)$$

$$y = \int \theta\, dx \quad \text{...III}(e)$$

On the axis of the member, the structural member is divided into several panels or segments by points called panel points or nodal points. This can be done for the sake of numerical integration. But for the sake of conveniene, generally the panels of equal lengths are preferred. While starting the process of numerical integration, the loads are replaced by their statically equivalent forces acting at the panel points. Few cases in which the bending moments at the panel points are known or can be determined easily, the numerical integration process may start with the bending moments or curvatures. The curvature over every panel may first be replaced by equivalent concentration at the panel points. For finding the curvature or equivalent load at the the panel points, any suitable distribution of loads or curvatures may be assumed. For this purpose, trapezoidal rule and the parabolic rule are commonly used.

1. Trapezoidal Rule : Here we take the x-axis along the centroidal axes of the segment and the function may represent the loading or the bending moment. It is assumed that the function $f(x)$ varies linearly with x. The variation of the function along segment AB of length h is represented by a trapezium as show in figure (3.8) below :

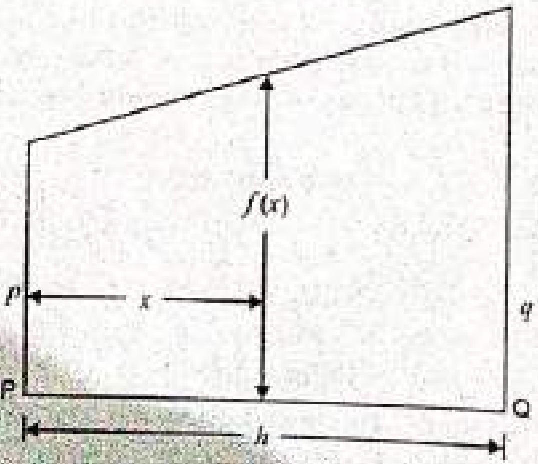

Figure 3.8

...the values of the function or the ordinates at P and Q are equal to p and q. The area of the trapezium is $\frac{p+q}{2}h$ and its centroid is located at a distance $\left(\frac{2p+q}{p+q}\cdot\frac{h}{3}\right)$ from Q. The trapezoidal area may be replaced by two equivalent areas $\overline{\text{P}}$ and $\overline{\text{Q}}$ concentrated at the points P and Q as shown in the figure (3.9) below:

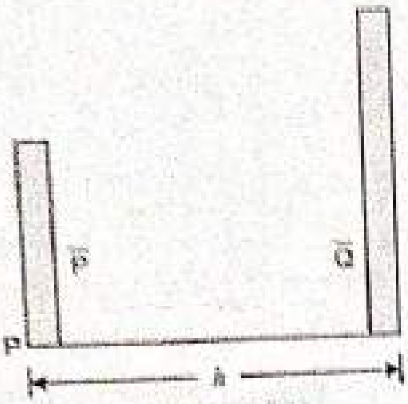

Figure 3.9

Subject to the conditions that the area of the trapezium is equal to the sum of $\overline{\text{P}}$ and $\overline{\text{Q}}$ and the the centroid of the trapezium coincides with the centroid of $\overline{\text{P}}$ and $\overline{\text{Q}}$. These conditions may be expressed by the following equations :

$$0.5\,(p+q)\,h = \overline{\text{P}} + \overline{\text{Q}} \qquad \text{...IV}(a)$$

$$\frac{2p+q}{p+q}\cdot\frac{h}{3} = \frac{\overline{\text{P}}h}{\overline{\text{P}}+\overline{\text{Q}}} \qquad \text{...IV}(b)$$

From these equations, the equivalent concentrations are given by

$$\overline{\text{P}} = \frac{h}{6}(2p+q) \qquad \text{...V}(a)$$

$$\overline{\text{Q}} = \frac{h}{6}(2q+p) \qquad \text{...V}(b)$$

2. Parabolic Rule : Here we assumed that the variation of the function is parabolic. Consider two adjacent segments PQ and QR each of length "h" as shown in figure (3.10) because parabola is completely defined only when at least three of its ordinates are given.

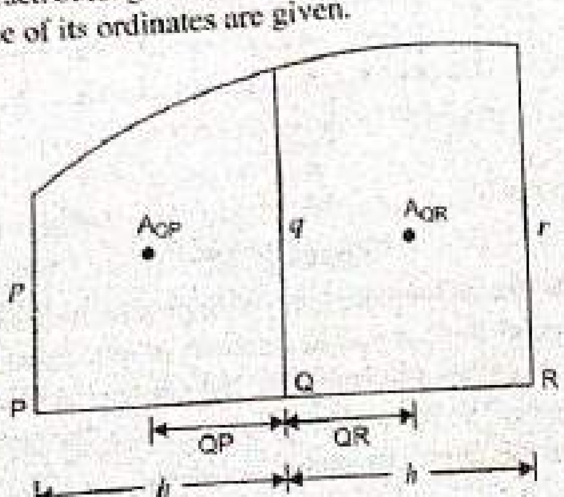

Figure 3.10

Let the values of the function or the ordinates at P, Q and R be equal to p, q and r and areas under the curve in panels PQ and QR be A_{QP} and A_{QR} respectively. The equivalent concentrations $\bar{P}$ and $\bar{Q}_{QP}$ at P and Q for the panel PQ and $\bar{Q}_{QR}$ and $\bar{R}$ at Q and R for the panel QR as shown in figure (3.11).

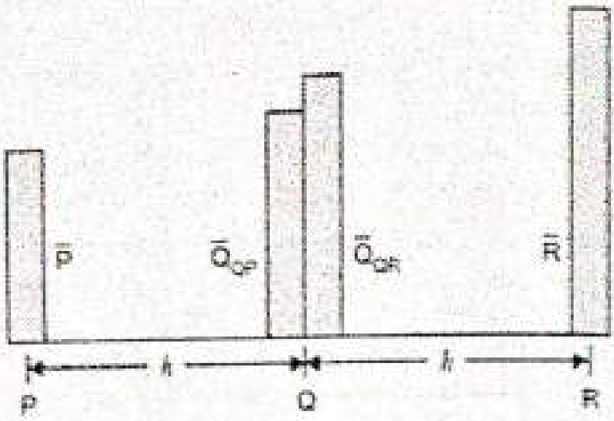

Figure 3.11

May be determined from the following equations :

$$\bar{P} + \bar{Q}_{QP} = A_{QP} \quad \text{...V(a)}$$

$$\bar{Q}_{QR} + \bar{R} = A_{QR} \quad \text{...V(b)}$$

$$A_{QP}\ \bar{y}_{QP} = \bar{P}h \quad \text{...V(c)}$$

$$A_{QR}\ \bar{y}_{QR} = \bar{R}h \quad \text{...V(d)}$$

Where A_{QP}, A_{QR} are areas in panels QP and QR as shown in figure (10) and $\bar{y}_{QP}$, $\bar{y}_{QR}$ are distances of centroids of A_{QP} and A_{QR} from Q and shown in figure (3.11).

Following are the steps according to which the process of successive numerical integration may be carried out. Procedure is explained with the help of the simply supported beam as shown in the figure (3.12) below :

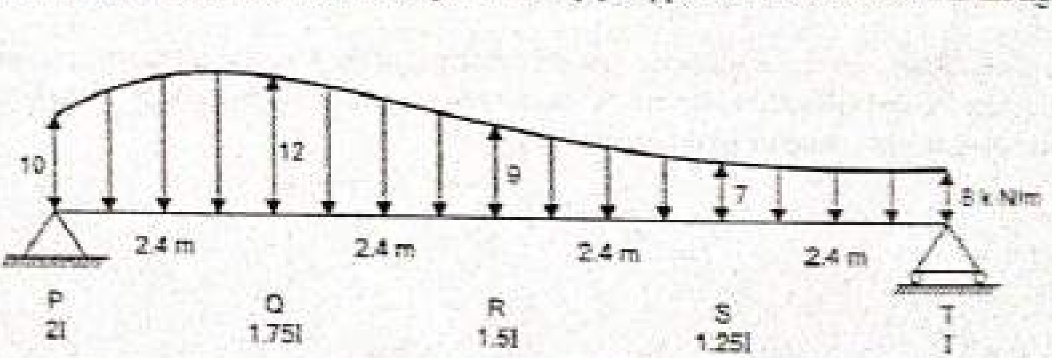

Figure 3.12

Step 1 : Divide the beam in to a convenient number of panels. Here in above example, it is divided in to four panels each having a length $h = 2.4$ m. The accuracy of numerical integration increases with the number of panels but it also increases the computations. The moment of inertia decreases linearly from 2I at P to I at Q as shown in the figure. The intensities of loading W at the panel points P, Q, R, S and T are equal to 10, 12, 9, 7 and k N/m respectively as shown in figure.

Step II : Determine the load concentrations $\overline{W}$ at all panel points using either the trapezoidal rule or the parabolic rule.

When the trapezoidal rule is used, equations

$$\overline{P} = \frac{h}{6}(2p + q)$$

and

$$\overline{Q} = \frac{h}{6}(2q + p)$$

give the values of the equivalent load concentrations $\overline{W}$ at each end of a panel with $\frac{h}{6}$ as the common factor. The net equivalent load concentrations $\overline{W}_n$ at the interior panel points Q, R and S are obtained by adding the two concentrations. When the parabolic rule is used, there are two alternative approaches. In the first approach, firstly two adjacent panels PQ and QR are taken and the equivalent load concentrations $\overline{W}$ are computed at the ends of panels. PQ and QR from equation.

$$\overline{P} = \frac{h}{24}(7p + 6q - r) \quad \text{...VII}(a)$$

$$\overline{Q}_{QP} = \frac{h}{24}(3p + 10q - r) \quad \text{...VII}(b)$$

$$\overline{Q}_{QR} = \frac{h}{24}(3r + 10q - p) \quad \text{...VII}(c)$$

$$\overline{R} = \frac{h}{24}(7r + 6q - p) \quad \text{...VII}(d)$$

Adding $\overline{Q}_{QP}$ and $\overline{Q}_{QR}$, the equivalent concentration at panel point B,

$$\overline{Q} = \frac{h}{24}(2p + 20q + 2r) \quad \text{...VII}(e)$$

It should be noted that the areas A_{QP} and A_{QR} and the position of their centroids may be determined by representing the parabolic curve by the following equation.

$$f(x) = a_1 + a_2 x + a_3 x^2 \quad \text{...VIII}$$

where the constants a_1, a_2 and a_3 may be determined by noting that the values of $f(x)$ at P, Q and R are p, q and r.

Note : Determine the areas A_{QP} and A_{QR} and their centroids from equation (VIII), put in equation (VI) and solving for the equivalent concentrations, we get the set of equation VII.

Now again the panels RS and ST are treated in a similar manner. The values obtained are shown in the table with common factor as $\frac{h}{24}$. The two values at panel points Q, R and S are added, giving the net equivalent load concentrations. Below we give the bending moment diagram, shear force diagram and deflection curve for the giiven beam, figure (3.12).

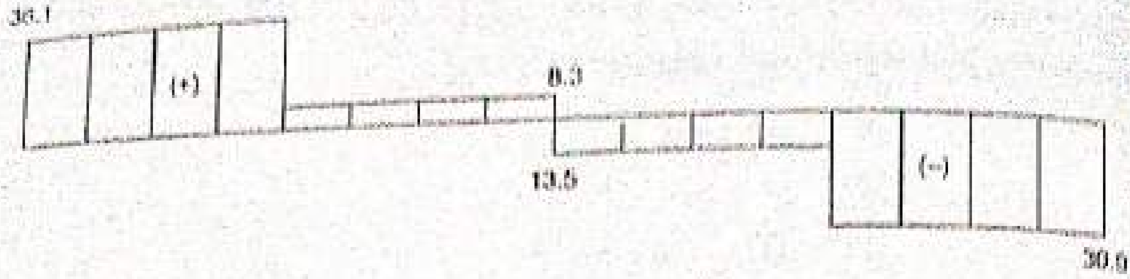

Figure 3.13 : Shear Force Diagram

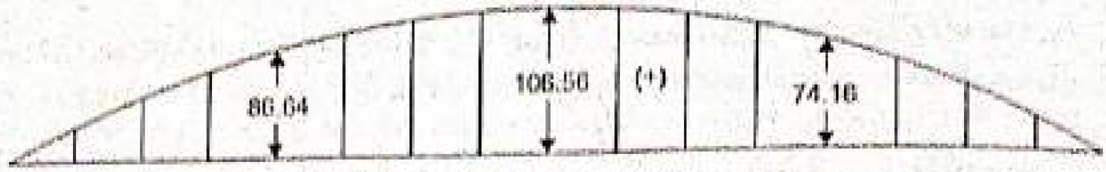

Figure 3.14 : Bending Moment

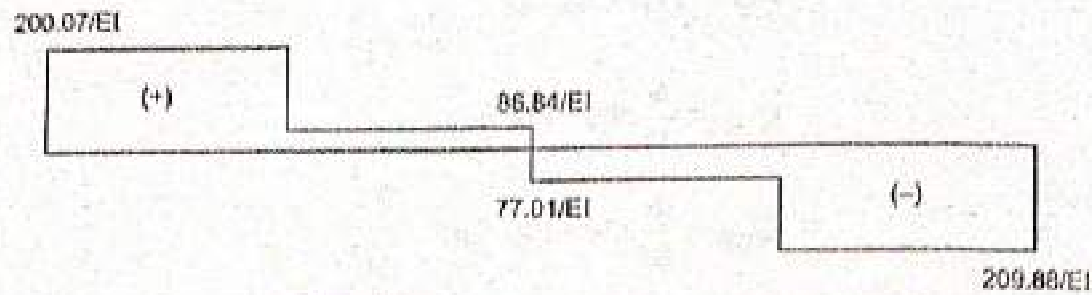

Figure 3.15 : Panel Slope Diagram

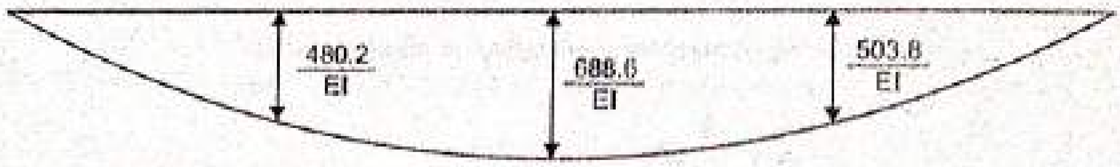

Figure 3.16 : Deflection Curve

In the second approach, the load concentrations at the extreme panel points P and T are calculated from equations VII(*a*) and VII(*d*) respectively and those at the interior points Q, R and S are calculated from equations VII(*b*) and VII(*c*). The values obtained are with common factor $\frac{h}{24}$. The two values at panel points Q, R and S are added algebrically giving the net Equivalent load concentrations $\overline{W}_a$. The second apporach is generally believed to give better results. These values are being used for further calculations.

Step III : Determine the vertical reaction V_P at P by taking moments about E of the equivalent load concentrations $\overline{W}$

$$V_P \times 9.6 - 133\left(\frac{h}{24}\right) \times 9.6 - (141 + 137) \times \frac{h}{24} \times 7.2$$

$$-(119 + 99) \times \frac{h}{24} \times 4.8 - (89 + 85) \times \frac{h}{24} \times 2.4 = 0$$

or

$$V_P = 494\left(\frac{h}{24}\right)$$

From the vertical equilibrium of forces,

$$V_T = (133 + 141 + 137 + 119 + 99 + 89 + 85 + 89)\,\frac{h}{24} - 494 \times \frac{h}{24}$$

$$= 398\left(\frac{h}{24}\right)$$

The value of V_A may be assumed arbitrarily and corrected latter by means of a linear correction to the bending moments as explained in step (V).

Step IV : Determine the shear force in each of the four panels. The shear force is constant over the entire length of the panel because the actual distribution load has been replaced by the equivalent concentrated loads. The shear force is called panel shear and denoted by "Q_P".

$$Q_{P_{PQ}} = V_A + \overline{W}_{PQ}$$

$$= 494\left(\frac{h}{24}\right) - 133\left(\frac{h}{24}\right) = 361\left(\frac{h}{24}\right)$$

$$Q_{P_{QR}} = Q_{P_{PQ}} + \overline{W}_{nQ}$$

$$= 361\left(\frac{h}{24}\right) - 278\left(\frac{h}{24}\right) = 83\left(\frac{h}{24}\right)$$

$$Q_{P_{RS}} = Q_{P_{QR}} + \overline{W}_{nR}$$

$$= 83\left(\frac{h}{24}\right) - 218\left(\frac{h}{24}\right) = -135\left(\frac{h}{24}\right)$$

$$Q_{P_{ST}} = Q_{P_{RS}} + \overline{W}_{nS}$$

$$= -135\left(\frac{h}{24}\right) - 174\left(\frac{h}{24}\right) = -309\left(\frac{h}{24}\right)$$

These panel shears are plotted in figure (13) to obtain the panel shear diagram or shear force diagram. The panel diagram or shear force diagram. The panel shears are shown in the respective panels. The shear force diagram comprises of a series of rectangles. Panel shears are shown in the table also.

Step V : From the panel shears shown in figure and table, find the bending moments at the panel points.

The bending moment at any panel point is equal to the area of the panel shear diagram on the left side of the panel point.

$$M_P = 0$$

$$M_Q = M_P + Q_{P_{PQ}} \times h$$

$$= 0 + 361\left(\frac{h}{25}\right) \times h = 361\left(\frac{h^2}{24}\right).$$

$$M_R = M_Q + Q_{P_{QR}} \times h$$

$$= 361\left(\frac{h^2}{24}\right) + 83\left(\frac{h}{24}\right) \times h = 444\left(\frac{h^2}{24}\right).$$

$$M_S = M_R + Q_{P_{RS}} \times h$$

$$= 444\left(\frac{h^2}{24}\right) - 135\left(\frac{h}{24}\right) \times h = 309\left(\frac{h^2}{24}\right).$$

$$M_T = M_S + Q_{P_{ST}} \times h$$

$$= 309\left(\frac{h^2}{24}\right) - 309\left(\frac{h}{24}\right) \times h = 0.$$

These values are used to draw the bending moment diagram as shown in figure (14). values are also shown in the table.

The bending moment at T is found to be zero because the correct value of V_A was used in step (IV). Alternatively V_A may be assumed arbitrarily to be equal to $300\left(\frac{h}{24}\right)$. When this value of V_A is used, the arbitrary panel shears Q_P' are show in the table. Further, taking the bending moment at A to be zero, the arbitrary bending moments at other panel points M′ may be determined by proceeding from left to right and successively adding the areas of the arbitary panel shear diagram as shown in the table. The bending moment M_E' at T thus works out to be $-244\left(\frac{h^2}{24}\right)$. Since the bending moment at T must be zero, a correction of $244\left(\frac{h^2}{24}\right)$ should be applied at T. The correcting moments M″ vary linearly from $244\left(\frac{h^2}{24}\right)$ at T to zero at A as shown in table. The linear correction of moments is equivalent so rotating the arbitrary or un corrected bending moment diagram such that the condition of zero moments at both ends is satisfied. The correct values of the bending moment shown in table are obtained by adding M′ and M″ given in table colums.

Step VI : Using equation $\phi = -\frac{M}{EI}$ and the values of the bending moment from the table, determine the curvature ϕ represents the intensity of load W* on the conjugate beam. Therefore, the curvature diagram of the original beam represents the load diagram of the conjugate beam. Here "*" symbol on "W" is used to distinguish the conjugate beam from the original beam.

...Step VII : Using the values of W*, find the equivalent curvature concentrations $\bar{\phi}$ at the panel ...in the same manner as the equivalent load concentrations in step II. Using equations VII(a) and ...for the panel points P and T and equations VII(b) and VII(c) for the panel points Q, R and S the ...equivalent curvature concentrations $\bar{\phi}$ are shown in table. The net equivalent curvature concentrations ...obtained by adding the two values at panel points Q, R and S. Note that this step is similar to step 2.

...note that $\bar{\phi}$ and $\bar{\phi}_n$ for the original beam are equivalent to $\overline{W}*$ and $\overline{W}_n^*$ for the conjugate beam.

Step VIII : Determine the end slopes Q_A at end P by taking moments about T of the equivalent ...curvature concentrations $\bar{\phi}$.

$$\text{EI}\, Q_P \times 9.6 - 941.8\left(\frac{h^3}{576\text{ EI}}\right)\times 9.6$$

$$-(1767+2951)\left(\frac{h^3}{576\text{ EI}}\right)\times 7.2$$

$$-(3331.7-3495.3)\left(\frac{h^3}{576\text{ EI}}\right)\times 4.8$$

$$-(3361+2176)\left(\frac{h^3}{576\text{ EI}}\right)\times 2.4 = 0$$

or

$$\theta_P = 9277.8\left(\frac{h^3}{576\text{ EI}}\right)$$

Alternatively, the value of θ_P may be assumed arbitrarily and later corrected by calculating the arbitrary or uncorrected values θ'_P, y' and y similar to step V. When the concept of conjugate beam is used, this step involues the determination of the support reaction V_P^* and P in the conjugate bema.

Step IX : From the end slope θ_P and the equivalent curvature concentrations $\bar{\phi}$, find the panel slopes θ_P.

$$\theta_{P_{PQ}} = \theta_P + \bar{\phi}_{PQ}$$

$$= 9277.8\left(\frac{h^3}{576\text{ EI}}\right) - 941.8\left(\frac{h^3}{576\text{ EI}}\right)$$

$$= 8336\left(\frac{h^3}{576\text{ EI}}\right)$$

$$\theta_{P_{QR}} = \theta_{P_{PQ}} + \bar{\phi}_{nQ}$$

$$= 8336\left(\frac{h^3}{576\ EI}\right) - 4718\left(\frac{h^3}{576\ EI}\right) = 3618\left(\frac{h^3}{576\ EI}\right)$$

$$\theta_{P_{RS}} = \theta_{P_{QR}} + \bar{\phi}_{nR}$$

$$= -3209\left(\frac{h^3}{576\ EI}\right)$$

$$\theta_{P_{ST}} = \theta_{P_{RS}} + \bar{\phi}_{nS}$$

$$= -8745\left(\frac{h^3}{576\ EI}\right)$$

These panel slopes are used to draw the panel slope diagram as shown in figure (3.15). Note that this step is similar to step IV and that the panel slope θ_P in the original beam is analogous to panel shear θ_P^* in the conjugate beam.

Step X : From the panel slopes θ_P, find the deflections y at the panel points. Since the deflection at any point is equal to the area of the panel slope diagram on the left hand side of the panel point.

$$y_Q = 8386\left(\frac{h^4}{576\ EI}\right)$$

$$y_R = 11954\left(\frac{h^4}{576\ EI}\right)$$

$$y_S = 8745\left(\frac{h^4}{576\ EI}\right)$$

These values are used to draw the deflection curve figure (3.16). It should be noted that the determination of deflection y from curvature ϕ in the original beam is analogous to the determination of bending moment M* from the intensity of loading W* in the conjugate beam.

In order to draw the shear force diagram, the values of the shear force at the panel point Q may be determined from the equivalent laod concentrations.

$$Q_P = V_A = 494\left(\frac{h}{24}\right)$$

$$Q_Q = Q_{PQ} + \overline{W}_{QP}$$

$$= 361\left(\frac{h}{24}\right) - 141\left(\frac{h}{24}\right) = 220\left(\frac{h}{24}\right)$$

$$Q_R = V_{QR} + \overline{W}_{RQ}$$

$$= 83\left(\frac{h}{24}\right) - 119\left(\frac{h}{24}\right) = -36\left(\frac{h}{24}\right)$$

$$Q_S = Q_{RS} + \overline{W}_{SR}$$

$$= -135\left(\frac{h}{24}\right) - 89\left(\frac{h}{24}\right) = -224\left(\frac{h}{24}\right)$$

$$Q_T = Q_{ST} + \overline{W}_{TS}$$

$$= -309\left(\frac{h}{24}\right) - 89\left(\frac{h}{24}\right) = -398\left(\frac{h}{24}\right)$$

These values are used to draw the shear force diagram. The slopes at the panel points may be obtained in similar manner using the equivalent curvature concentrations and panel slopes.

For Example

$$\theta_Q = \theta_{PQ} + \bar{\phi}_{QP}$$

$$= 8336\left(\frac{h^3}{576\ EI}\right) - 1767\left(\frac{h^3}{576\ EI}\right) = 6569\left(\frac{h^3}{576\ EI}\right)$$

In various examples, in the calculations which are arranged in tabular form, the Kilo-Newton and metre are used as the units of force and length respectively. Quantities in the tables are represented by the following symbols.

I = moment of inertia.

ω = intensity of load taken positive when upward.

$\bar{\omega}$ = equivalent load concentrations.

$\bar{\omega}_n$ = net equivalent load concentrations at panel points.

Q'_P = arbitrary, trial or uncorrected panel shears.

M' = arbitrary, trial or uncorrected bending moment at panel points.

M'' = linear corrections to arbitrary, trial or uncorrected bending moment at panel points.

M = corrected bending moment at panel points.

Q_P = corrected panel shear.

Q = shear force of the panel points.

$\phi = -\frac{M}{EI}$ = curvature at panel points in the original beam.

= intensity of load ω^* in the conjugate beam.

$\bar{\phi}$ = equivalent curvature concentration at panel points in the original beam.

= equivalent load concentration $\bar{\omega}^*$ in the conjugate beam.

$\bar{\phi}_n$ = net equivalent curvature concentration at panel points in original beam.

= net equivalent load concentration $\bar{\omega}_n^*$ in the conjugate beam.

ϕ'_P = arbitrary, trial or uncorrected panel slopes in original beam.

= arbitrary, trial or uncorrected panel sheares $Q_P'^*$ in the conjugate beam.

Q6 Given D.E $3\frac{d^2y}{dx^2} - \frac{dy}{dx} + 8 = 0$ — (*)

find $y(0.3)$ & $y(0.6)$ using Galerkin's Method

Boundary cond^n $y(0)=1$ | $y(1)=2$

means $x_0=0, y=1$ | $x=1, y=2$

Sol^n order $(n)=2$

poly $(n+1)=3$

means constants c_0, c_1, c_2, c_3 & Parameter $(n+2)=4$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \quad — (1)$$

using B.cond put $y(0)=1 \Rightarrow x=0, y=1$

$$\boxed{1 = c_0} \quad — (2)$$

using 2nd B.cond^n $y(1)=2$ in (1)

$$2 = c_0 + c_1 + c_2 + c_3$$

$$2 = 1 + c_1 + c_2 + c_3 \Rightarrow \boxed{c_1 + c_2 + c_3 = 1}$$

$$\Rightarrow \boxed{c_1 = -c_2 - c_3 + 1} \quad — (3)$$

from (1) : $y = 1 + [-c_2 - c_3 + 1]x + c_2 x^2 + c_3 x^3$

$$y = 1 - c_2 x - c_3 x + x + c_2 x^2 + c_3 x^3$$

$$y = c_2[-x + x^2] + c_3[-x + x^3] + x + 1$$

$$\frac{dy}{dx} = c_2[-1 + 2x] + c_3[-1 + 3x^2] + 1$$

$$\frac{d^2y}{dx^2} = c_2[2] + c_3[6x]$$

Put dy/dx & $\frac{d^2y}{dx^2}$ in eqn (*)

$$3\frac{d^2y}{dx^2} - \frac{dy}{dx} + 8 = 0$$

Residue $(R) = 3[2c_2 + 6xc_3] - [c_2(-1+2x) + c_3[-1+3x^2] + 1] + 8$

$R = 6c_2 + 18c_3x - c_2(-1+2x) - c_3(-1+3x^2) - 1 + 8$

~~$R = c_2(6+1$~~

$R = 6c_2 + 18c_3x + c_2 - 2c_2x + c_3 - 3c_3x^2 - 1 + 8$

$R = c_2(6+1-2x) + c_3(18x+1-3x^2) + 7$

$R = c_2(7-2x) + c_3(18x+1-3x^2) + 7$.

The weighted integral form

$(0<x<1)$ $\int_0^1 w_i \cdot R\,dx = 0$

Since $y = c_2\underbrace{[-x+x^2]}_{w_1} + c_3\underbrace{[-x+x^3]}_{w_2} + x + 1$

$\int_0^1 (-x+x^2)\cdot[c_2(7-2x) + c_3(18x+1-3x^2) + 7]\,dx = 0$

$\int_0^1 [c_2(7-2x)(-x+x^2) + c_3(18x+1-3x^2)(-x+x^2) + 7(-x+x^2)]\,dx = 0$

$\int_0^1 c_2[-7x + 7x^2 + 2x^2 - 2x^3] + c_3(-18x^2 + 18x^3 - x + x^2 + 3x^3 - 3x^4) + (-7x + 7x^2)\,dx = 0$

$\int_0^1 c_2[-7x + 9x^2 - 2x^3] + c_3[-17x^2 + 21x^3 - x - 3x^4] + 7(-x+x^2)]\,dx = 0$

$\Rightarrow c_2\left[\frac{-7x^2}{2} + \frac{9x^3}{3} - \frac{2x^4}{4}\right]_0^1 + c_3\left[\frac{-17x^3}{3} + \frac{21x^4}{4} - \frac{x^2}{2} - \frac{3x^5}{5}\right]_0^1 + 7\left[\frac{-x^2}{2} + \frac{x^3}{3}\right]_0^1 = 0$

$c_2\left[\frac{-7}{2} + 3 - \frac{1}{2}\right] + c_3\left[\frac{-17}{3} + \frac{21}{4} - \frac{1}{2} - \frac{3}{5}\right] + 7\left(\frac{-1}{2} + \frac{1}{3}\right) = 0$

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$$c_2\left[\frac{-7+6-1}{2}\right]+c_3\left[\frac{-340+315+30-36}{60}\right]+7\left[\frac{-3+2}{6}\right]=0$$

$$\cancel{c_2(-1)+c_3[-2.0166]+7(0.1666)=0}$$

$$c_2(-1)+c_3(-1.5166)+7(-0.1666)=0$$

$$\Rightarrow \boxed{c_2(-1)+c_3(-1.5166)=1.1662} \quad \text{—— (4)}$$

& $W_2=-x+x^3$

$$\int_0^1 W_2 \cdot R\, dx=\int_0^1(-x+x^3)\left[c_2(7-2x)+c_3(18x+1-3x^2)+7\right]dx$$

On simplyfying we get

Try it

$$\boxed{c_2(-1.4833)+c_3(-2.4)=1.75} \quad \text{—— (5)}$$

On solving (4) & (5) we get

$$\boxed{c_2=-0.9706} \qquad \boxed{c_3=-0.12928}$$

Since $y=c_2(-x+x^2)+c_3(-x+x^3)+x+1$

$$y=(-0.9706)(-x+x^2)+(-0.12928)(-x+x^3)+1$$

1. $y(0.3)=1.53911$ ($x=0.3$)

2. $y(0.6)=1.8825$ ($x=0.6$) Ans

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Q. Solve the boundary Value Problem defined by Galerkin's method.

Soln: $y'' = x+y \quad y(0)=0,\ y(1)=0$ — (*)

order(n) = 2 $\quad 0 \to 1 \quad \int_0^1$

Poly(n+1) = 3

Const(n+2) = 4

$\therefore \quad y = c_0 + c_1x + c_2x^2 + c_3x^3$ — ①

using $y(0)=0$ & $y(1)=0$

$\boxed{0 = c_0}$ — ② $\quad 0 = c_0 + c_1 + c_2 + c_3$

$\Rightarrow \boxed{0 = c_1 + c_2 + c_3}$

$\Rightarrow \boxed{c_1 = -c_2 - c_3}$ — ③

from ① $\quad y = (-c_2 - c_3)x + c_2x^2 + c_3x^3$

$y = c_2(\underbrace{-x+x^2}_{w_1}) + c_3(\underbrace{-x+x^3}_{w_2})$

$\frac{dy}{dx} = c_2(-1+2x) + c_3(-1+3x^2)$

$\frac{d^2y}{dx^2} = c_2(2) + c_3(6x)$

Put dy/dx & $\frac{d^2y}{dx^2}$ in eqn (*)

$y'' = x + y$

$R = 2c_2 + c_3(6x) - x - c_2(-x+x^2) + c_3(-x+x^3)$

~~$R = c_2(2+x) + c_3(6x$~~

$R = c_2(2+x-x^2) + c_3(6x-x+x^3) - x$

$\int_0^1 w_1 R\,dx = 0 = \int_0^1 (-x+x^2)\left[c_2(2+x-x^2) + c_3(5x+x^3) - x\right]dx = 0$

$\Rightarrow \int_0^1 \left[c_2(-x+x^2)(2+x-x^2) + c_3(-x+x^2)(5x+x^3) - x(-x+x^2)\right]dx = 0$

$\Rightarrow \int_0^1 \left[c_2\left(-2x - x^2 + x^3 + 2x^2 + x^3 - x^4\right) + c_3\left[-5x^2 - x^4 + 5x^3 + x^5\right] + x^2 - x^3\right]dx = 0$

$\Rightarrow \int_0^1 c_2\left[2x^3 + x^2 - x^4 - 2x\right] + c_3\left[x^5 - x^4 + 5x^3 - 5x^2\right] + x^2 - x^3\,dx = 0$

$\Rightarrow \left[c_2\left[\frac{2x^4}{4} + \frac{x^3}{3} - \frac{x^5}{5} - 2\frac{x^2}{2}\right] + c_3\left(\frac{x^6}{6} - \frac{x^5}{5} + \frac{5x^4}{4} - \frac{5x^3}{3}\right] + \frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 0$

$\Rightarrow c_2\left[\frac{1}{2} + \frac{1}{3} - \frac{1}{5} - 1\right] + c_3\left[\frac{1}{6} - \frac{1}{5} + \frac{5}{4} - \frac{5}{3}\right] + \left(\frac{1}{3} - \frac{1}{4}\right) = 0$

$\Rightarrow \boxed{c_2[-0.3666] + c_3[-0.45] + (0.0833) = 0}$ —— (4)

Now $\int_0^1 w_2 \cdot R\, dx$

$\Rightarrow \int_0^1 (-x+x^3)\left[c_2(2+x-x^2) + c_3(5x+x^3) - x\right]dx = 0$

$\Rightarrow \int_0^1 \left[c_2(-x+x^3)(2+x-x^2) + c_3(-x+x^3)(5x+x^3) - x(-x+x^3)\right]dx$

$\Rightarrow \int_0^1 c_2\left[-2x - x^2 + x^3 + 2x^3 + x^4 - x^5\right] + c_3\left[-5x^2 - x^4 + 5x^4 + x^6\right] + (x^2 - x^4)\,dx$

$\Rightarrow \int_0^1 c_2\left[-x^5 + x^4 + 3x^3 - x^2 - 2x\right] + c_3\left[x^6 + 4x^4 - 5x^2\right] + (x^2 - x^4)\,dx$

$\Rightarrow c_2\left(-\frac{x^6}{6} + \frac{x^5}{5} + \frac{3x^4}{4} - \frac{x^3}{3} - \frac{2x^2}{2}\right)_0^1 + c_3\left(\frac{x^7}{7} + \frac{4x^5}{5} - \frac{5x^3}{3}\right)_0^1 + \left(\frac{x^3}{3} - \frac{x^5}{5}\right)_0^1$

$$c_2\left[\frac{-1}{6}+\frac{1}{5}+\frac{3}{4}-\frac{1}{3}-1\right]+c_3\left[\frac{1}{7}+\frac{4}{5}-\frac{5}{3}\right]+\left[\frac{1}{3}-\frac{1}{5}\right]=0$$

$$\therefore \boxed{c_2(-0.55)+c_3(-0.7238)+(0.1333)=0} \quad \text{— (5)}$$

on Solving (4) & (5)

$$\begin{aligned} -0.3666\,c_2 - 0.45\,c_3 &= -0.0833 \quad \times 0.55 \\ -0.55\,c_2 - 0.7238\,c_3 &= -0.1333 \quad \times 0.3666 \end{aligned}$$

$$\begin{aligned} -0.20163\,c_2 - 0.2475\,c_3 &= -0.045815 \\ -0.20163\,c_2 - 0.26534\,c_3 &= -0.04886 \\ + \qquad\quad + \qquad\quad & \quad + \end{aligned}$$

$$+0.01784\,c_3 = 0.003045$$

$$\Rightarrow \boxed{c_3 = 0.17068}$$

Put c_3 in (4) $\quad -0.3666\,c_2 - 0.45(0.17068) = -0.0833$

$$\Rightarrow -0.3666\,c_2 - 0.076806 = -0.0833$$

$$\Rightarrow +0.3666\,c_2 = +0.006494$$

$$\Rightarrow \boxed{c_2 = 0.017714}$$

$$\therefore y = c_2(-x+x^2)+c_3(-x+x^3)$$

$$y = 0.017714(-x+x^2)+0.17068(-x+x^3)$$

NMCE

* Solution of system of linear equation.

$$2x+3y+z=0$$

Linear eq.:-

Equation which have one power.

Gauss Elimination method.

- Write the eq. in matrix notation.

$$AX=B$$

where $A=\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B=\begin{bmatrix} - \\ - \\ - \end{bmatrix}$.

- Augmental Matrix

$$[A:B]$$

$$\left[\begin{array}{ccc:c} - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{array}\right]$$

- Drow the upper trangular with help of Row operation

Ex:- $x+2y+3z=6$

$x+2y+z=4$

$x-y+2z=2$

Matrix notation

$AX=B$

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, B=\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}.$$

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Argumented Matrix.

$[A:B]$

$$\begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 1 & 2 & 1 & : & 4 \\ 1 & -1 & 2 & : & 2 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & 0 & -2 & : & -2 \\ 0 & -3 & -1 & : & -4 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -3 & -1 & : & -4 \\ 0 & 0 & -2 & : & -2 \end{bmatrix}$$

$Ax = B$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -1 \\ 0 & 0 & -2 \end{bmatrix}, x \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 6 \\ -4 \\ -2 \end{bmatrix}$$

$x + 2y + 3z = 6$

$-3y - z = -4$

$\cancel{-}2z = -\cancel{2}$

$z = 1$

$-3y - 1 = -4$

$-3y = -4 + 1$

$+3y = +3$

$y = 1$

$x + 2 + 3 = 6$

$x + 5 = 6$

$x = 6 - 5$

$x = 1$

$x = 1$

$y = 1$

$z = 1$.

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Gauss Jordon Method:-

- Convert the given matrix in the form of $Ax = B$.
- And solve for the Identical matrix with the help of Row operation

Ex:- $2x + 2y + 4z = 18$

$x + 3y + 2z = 13$

$3x + y + 3z = 14$

$Ax = B$.

$$\begin{bmatrix} 2 & 2 & 4 \\ 1 & 3 & 2 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 13 \\ 14 \end{bmatrix}.$$

$R_1/2$.

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 2 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 14 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, ~~$R_3 \to R_3$~~ $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ -13 \end{bmatrix}.$$

$R_2/2$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ -13 \end{bmatrix}$$

$R_3/3$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0.33 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 4.66 \end{bmatrix}$$

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$R_3 \leftrightarrow R_3$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ -15 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ -9 \end{bmatrix}$$

$R_3 / -3$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}$$

$R_1 \to R_1 + 2R_3$

$$\cancel{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6.34 \\ 2 \\ -0.33 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$x = 1 \quad y = 2 \quad z = 3$. Ans

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Gauss - Seidal Iteration Method.

1st check

- Diagonally dominated.
- if. Now, the system is diagonally dominated, then we can start Gauss seidel iteration method.
- Take $y_0 = z_0 = 0$ as initial approximate - if not mention in question.

Ex:- $20x + y - 2z = 17$

$3x + 20y - z = -18$

$2x - 3y + 20z = 25.$

DT $|20| > |1| + |-2|$

$|20| > |3| + |-1|$

$|20| > |2| + |-3|.$

The system of eqn is diagonally dominated. and Now, we can start Gauss seidel Iteration

Take ∴ The above system can be written as.

$$x = \frac{1}{20}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z)$$

$$z = \frac{1}{20}(25 - 2x + 3y).$$

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Let the Initial approximation be.

$x_0 = 0,\ y_0 = 0\ z_0 = 0.$

1^{st} Itn:-

$x_1 = \frac{1}{20}(17 - 0 + 2(0)) = \frac{17}{20} = 0.85$

$y_1 = \frac{1}{20}(-18 - 3(0.85) + 0) = -1.0275$

$z_1 = \frac{1}{20}(25 - 2(0.85) + 3(-1.0275)) = 1.0109.$

II^{nd} Itn.

$x_2 = \frac{1}{20}(17 + 1.0275 + 2(1.0109)) = 1.0024$

$y_2 = \frac{1}{20}(-18 - 3(1.0024) + 1.0109) = -0.9998.$

$z_2 = \frac{1}{20}(25 - 2(1.0024) + 3(-0.9998)) = 0.9999$

III^{rd} Itn:-

$x_3 = \frac{1}{20}(17 + 0.9998 + 2(0.9999)) = 0.9999$

$y_3 = \frac{1}{20}(-18 - 3(0.9999) + 0.9999) = -0.9999$

$z_3 = \frac{1}{20}(25 - 2(0.9999) + 3(-0.9999)) = 1.0000.$

The values in 2nd and 3rd Itn. are approximately equal, so we can stop here.

$\therefore x = 1,\ y = -1,\ z = 1$ is required Soluti

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Eigen value & eigen vector of a matrix:-

Let A be a square matrix of order n if there exist a no. λ and a non zero vector X such that $AX = \lambda$, than λ is eigen value and X is Eigen vector.

- $|A - \lambda I| = 0$ for eigen value
- $(A - \lambda I)(X) = 0$ for eigen vector.

Ex!- Find the eigen values and eigen vectors of the following matrices.

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

Sol^n:- For eigen value

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$-2-\lambda\left[(1-\lambda)(-\lambda)-(-2)(-6)\right] - 2(2\times-\lambda)+6-3\left(-2\times2-1(1-\lambda)\right] = 0$$

$-2-\lambda[(1-\lambda)(-\lambda)-(+6)\times(+2)]-2[2(-\lambda)-(+6)\times(+1)]-3[(2)(-2)-(1-\lambda)$

$-2-\lambda[-\lambda+\lambda^2-12]-2[-2\lambda-6]-3[-4-(-1+\lambda)]=0$

$-2-\lambda[-\lambda+\lambda^2-12]-2[-2\lambda-6]-3[-3-\lambda]=0$

$2\lambda-2\lambda^2+24+\lambda^2-\lambda^3+12\lambda+4\lambda+12+9+3\lambda=0$

$-\lambda^3-\lambda^2+21\lambda+45=0$

$\lambda^3+\lambda^2-21\lambda-45=0$

$\lambda=-3$

$$\begin{array}{r|rrrr} & 1 & 1 & -21 & -45 \\ -3 & & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$\lambda^2-2\lambda-15=0$

$\lambda^2-5\lambda+3\lambda-15=0$

$\lambda(\lambda-5)+3(\lambda-5)=0$

$(\lambda-5)=0,\ (\lambda+3)=0$

$\lambda=5,\ \lambda=-3$

$\lambda=-3,5,-3$

$\therefore$ Eigen value of $-3, 5, -3$ //

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{ for eigen vector.

$(A-\lambda I)(X)=0$

$$\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix} - \lambda\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\lambda = -3$

$$\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix} - \begin{bmatrix}-3 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}.$$

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\cancel{x+2y-3z=0}$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 + R_1$

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$x+2y-3z=0.$

$x=-2y+3z$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-2y+3z\\ y\\ z\end{bmatrix} = \begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}y + \begin{bmatrix}3\\ 0\\ 1\end{bmatrix}z.$$

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$\lambda = 5$

$(A - 5I)(X) = 0$

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_1/-1$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$R_2 \to R_2 - 2R_1, \; R_3 \to R_3 + 7R_1$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$R_3 \to R_3 + 2R_2.$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$x + 2y + 5z = 0$

$-8y - 16z = 0$

$z = 0$

$x + 2y = 0$

$x = -2y.$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -2y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

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Rayleigh's Power Method:-
(Largest & smallest eigen values and eigen vector).

- Let the initial eigen vector be $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ if not given in question.
- And start to solve the equation.

$$Y_1 = AX_0$$

Ex:- Find the largest eigen value and the corresponding eigen vector of the matrix.

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \text{ starting from } X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\underline{Y_2 = AX_0}$$

$$\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + 3 + 2 \\ 4 + 4 - 1 \\ 6 + 3 + 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 14 \end{bmatrix}$$

then we take the large value and devide with them.

$$\begin{bmatrix} 0 \\ 7 \\ 14 \end{bmatrix} = 14 \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix}$$

$$\lambda_1 = \underline{14}, \quad \underline{X_1} = \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix}.$$

λ

λ

$$Y_2 = AX_1$$

$$\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 - 1.5 + 2 \\ 0 + 2 - 1 \\ 0 + 1.5 + 5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \\ 6.5 \end{bmatrix}$$

$$6.5 \begin{bmatrix} 0.076 \\ 0.153 \\ 1 \end{bmatrix}$$

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$\lambda_2 = 6.5 \quad X_2 = \begin{bmatrix} 0.076 \\ 0.153 \\ 1 \end{bmatrix}$

$Y_3 = A X_2$

$$\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.076 \\ 0.153 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.076 - 0.459 + 2 \\ 0.304 + 0.612 - 1 \\ 0.456 + 0.459 + 5 \end{bmatrix} = \begin{bmatrix} 1.617 \\ -0.081 \\ 5.915 \end{bmatrix}$$

$$5.915 \begin{bmatrix} 0.273 \\ 0.014 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 5.915, \; X_3 = \begin{bmatrix} 0.273 \\ 0.014 \\ 1 \end{bmatrix}.$$

$Y_4 = A X_3$

$$\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.273 \\ 0.014 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.273 - 0.042 + 2 \\ 1.092 + 0.056 - 1 \\ 1.638 + 0.042 + 5 \end{bmatrix} \begin{bmatrix} 2.231 \\ 0.148 \\ 6.68 \end{bmatrix}$$

$$6.68 \begin{bmatrix} 0.333 \\ 0.022 \\ 1 \end{bmatrix}$$

$$\lambda_4 = 6.68, \; X_4 = \begin{bmatrix} 0.333 \\ 0.022 \\ 1 \end{bmatrix}$$

$$\text{error} = \frac{\lambda_4 - \lambda_3}{\lambda_4} \times 100 = \frac{6.68 - 5.915}{6.68} \times 100$$

$$= 11.45\%$$

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Matrix Inversion Method:-

- $A^{-1} = \frac{Adj(A)}{|A|}$
- $|A| \neq 0$ then A^{-1} exist
- $Adj(A) = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}^T$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Co factor of $a_{11} = (-1)^{1+1}(a_{22} \times a_{33} - a_{23} \times a_{32})$

$a_{12} = (-1)^{1+2}(a_{21} \times a_{33} - a_{23} \times a_{31})$ and so on....

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Solution of Algebric & Transcedntal Equation

* Algebraic equation:-

A polynomiayal eqn. $f(x) = a_0 x^{n-1} + a_1 x^{n-2} + a_2 x^{n-3} \ldots$

$a_{n-1} x + a_n = 0.$

e.g:- $f(x) = x^3 - x^2 + 3 = 0$

$f(x) = 3x^3 - x^2 + 1 = 0$

* Transcedental eqn:-

If eqn. $f(x) = 0$ Conteins fun. like trigonometric logrethmic, exponcial with eqn. is called transcedental eqn.

e.g = $xe^x - 1 = 0$

$\cos x - 2x - 3 = 0$

* Method for finding real root of an eqn.

Bisection Method:-

working rool:-

→ Let $f(x) = 0$ — (i) be the given eqn.

And a & b such that $f(a) < 0$ and $f(b) > 0$

Than 1st app.

$$x_1 = \frac{a+b}{2}$$

Examine its sign $f(x_1)$.

* If $f(x_1) < 0$. root lies b/w x_1^{-ve} & b^{+ve}.

$$x_2 = \frac{x_1 + b}{2}$$

Examine its sign.

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if $f(x_1) > 0$, Root lies b/w a & x_1

$$x_2 = \frac{x_1 + a}{2}$$ //

Ex:-

$$x^3 + x^2 + x + 7 = 0$$

Soln:-

Let $f(x) = x^3 + x^2 + x + 7$.

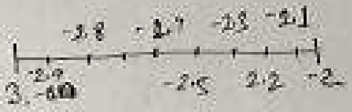
$f(0) = (0)^3 + (0)^2 + 0 + 7 = 7 > 0$

$f(-1) = (-1)^3 + (-1)^2 + (-1) + 7 = 6 > 0$

$f(-2) = (-2)^3 + (-2)^2 + (-2) + 7 = 1 > 0$

$f(-3) = (-3)^3 + (-3)^2 + (-3) + 7 = -14 < 0$

Roots lies b/w $\overset{+ve}{-2}, \overset{-ve}{-3}$



$f(-2.5) = (-2.5)^3 + (-2.5)^2 + (-2.5) + 7 = -4.876 < 0$

$f(-2.3) = (-2.3)^3 + (-2.3)^2 + (-2.3) + 7 = -2.177 < 0$

$f(-2.2) = (-2.2)^3 + (-2.2)^2 + (-2.2) + 7 = -8.048 < 0$

$f(-2.1) = (-2.1)^3 + (-2.1)^2 + (-2.1) + 7 = 0.049 > 0$

$\therefore$ root lies b/w (-2.1) & (-2.2).

1st App.

$$x_1 = \frac{a+b}{2} = \frac{-2.2 + (-2.1)}{2} = -2.15$$

$f(x_1) = f(-2.15) = (-2.15)^3 + (-2.15)^2 + (-2.15) + 7 = -0.46587$

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$\therefore$ Root lies b/w (-2.1) & (-2.15).

IInd app.

$$x_2 = \frac{x_1+b}{2} = \frac{(-2.15)+(-2.1)}{2} = -2.125$$

$$f(x_2) = f(-2.125) = (-2.125)^3 + (-2.125)^2 + (2.125) + 7 = -0.20507 < 0$$

$\therefore$ Root lies b/w (-2.125) & (-2.1).

IIIrd app.

$$x_3 = \frac{x_2+b}{2} = \frac{(-2.125)+(-2.1)}{2} = -2.1125$$

$$f(x_3) = f(-2.1125) = (-2.1125)^3 + (-2.1125)^2 + (-2.1125) + 7$$
$$= -0.07720 < 0.$$

$\therefore$ Root lies b/w (-2.1125) & (-2.1).

ivth app.

$$x_4 = \frac{x_3+b}{2} = \frac{(-2.1125)+(-2.1)}{2} = -2.10625$$

$$f(x_4) = f(-2.10625) = (-2.10625)^3 + (-2.10625)^2 + (-2.10625) + 7$$

$$= -0.0138 < 0$$

$\therefore$ Root lies b/w (-2.10625) & (2.1)

vth app.

$$x_5 = \frac{x_4+b}{2} = \frac{(-2.10625)+(-2.1)}{2} = -2.10312$$

$$f(x_5) = f(-2.10312) = (-2.10312)^3 + (-2.10312)^2 + (-2.10312) + 7 =$$
$$\Rightarrow 0.0176 > 0$$

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$\therefore$ Root lies b/w (-2.10312) & (-2.1062)

(vi) app.

$$x_6 = \frac{x_4 + x_5}{2} = \frac{(-2.15\ldots)+(-2.10312)}{2} = -2.12666$$

$$= \frac{(-2.10312)+(-2.10625)}{2} = -2.10468$$

$$f(x_6) = f(-2.10468) = (-2.10468)^3 + (-2.10468)^2 + (-2.10468) + 7$$

$$= 1.94341 \times 10^{-3}$$

$$= \frac{1.94341}{1000} = 0.00194370$$

$\therefore$ Root lies b/w (-2.10468) & (-2.10625)

vii) app.

$$x_7 = \frac{x_6 + x_4}{2} = \frac{(-2.10468)+(-2.10625)}{2}$$

$$= -2.10546$$

$$f(x_7) = f(-2.10546) = (-2.10546)^3 + (-2.10546)^2 + (-2.10546) + 7$$

$$= \frac{-5.92196}{1000} = -0.0059220$$

$\therefore$ Root lies b/w (-2.10468) & (-2.10546)

viiith app.

$$x_8 = \frac{(-2.10468)+(-2.10546)}{2} = -2.1050$$

Hence the root correct to three decimal place after compare (vii)th & (viii)th app.

$= -2.105$ Ans

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Regula falsi / Method of false position.

Formula:-

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$x_0 \to$ -ve root

$x_1 \to$ +ve root

Ex. $f(x) = xe^x - 2 = 0$.

$f(0) = 0e^0 - 2 = -2 < 0$

$f(1) = 1e^1 - 2 = 0.718 > 0$

$\therefore$ Root lies b/w 0 & 1.

0 0.1 0.4 0.7 0.9 1

$f(0.5) = 0.5e^{0.5} - 2 = -1.1756 < 0$

$f(0.6) = 0.6e^{0.6} - 2 = -0.90 < 0$

$f(0.7) = 0.7e^{0.7} - 2 = -0.59 < 0$

$f(0.8) = 0.8e^{0.8} - 2 = -0.21 < 0$

$f(0.9) = 0.9e^{0.9} - 2 = 0.21 > 0$

$\therefore$ Root lies b/w 0.8 & 0.9

$x_0 \to 0.8$

$x_1 = 0.9$.

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0.8(0.21) - 0.9(-0.21)}{(0.21) - (-0.21)}$$

$$= \frac{0.357}{0.42} = 0.85$$

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$f(x_0) = f(0.85) = 0.85e^{0.85} - 2 = -0.0113 < 0$

$x_0 = 0.85$

$\therefore$ Root lies b/w 0.85 & 0.9

IInd Ito.

$x_0 = 0.85$

$x_1 = 0.9$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{0.85(0.21) - 0.21(-0.0113)}{(0.21) - (-0.0113)}$$

$$= \frac{0.180873}{0.2213} = 0.8173$$

$f(x_2) = f(0.8173) = 0.8173e^{0.3173} - 2 = -0.1493 < 0$

$\therefore$ Root lies b/w. 0.8173 & 0.9

IIIrd Iton:- $x_0 = 0.8173$ $x_1 = 0.9$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{0.8173(0.21) - 0.21(-0.1493)}{(0.21) - (-0.1493)}$$

$$= \frac{0.202986}{0.3593}$$

So on.

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Secant Method.

$x_0 \to -ve$

$x_1 \to +ve$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}.$$

Ex:- Find the root of the eqn. $\sin x - x + 0.5 = 0$, using secant method to four decimal places.

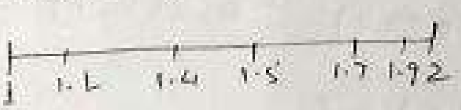
$f(x) = \sin x - x + 0.5$

$f(0) = 0.5 > 0$

$f(1) = \sin 1 - 1 + 0.5 = 0.34147 > 0$

$f(2) = \sin 2 - 2 + 0.5 = -1.09070 < 0$

$\therefore$ Root lies b/w 1 & 2



$f(1.5) = \sin 1.5 - 1.5 + 0.5 = -0.0025050 < 0$

$\therefore$ Root lies b/w 1 & 1.5.

$x_0 \to 1.5$

$x_1 \to 1.$

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3rd I.t.

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{1.5(0.34147) - \cancel{0.34147}\,1(-0.0025050)}{0.34147 - (-0.0025050)}$$

$$= \frac{\cancel{0.513060}}{\cancel{0.343475}} = \frac{0.51471}{0.343475}$$

$$= 1.49635$$

$f(x_2) = f(1.49156) = \sin 1.49165 - 1.49156 + 0.5$

$= -0.0053095.$

$\therefore$ Root lies b/w 1 & 1.49156.

$x_0 \to 1$ ~~1.49156~~ $x_1 = 1$

$x_2 \to$ $x_2 = 1.49635$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{1(-0.005309) - 1.49835(0.34147)}{(-0.005309) - (0.34147)}$$

$$= \frac{-0.516267}{-0.34677}$$

$$= 1.48879$$

$f(x_3) = f(1.48879) = \sin 1.48879 - 1.48879 + 0.5$

$= -0.007849.$

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Newton Raphson Method.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

where, $i = 0, 1, 2, 3, \ldots$

Ex:- Use Newton Raphson Method to obtain a root correct to three decimal places of the eqn.

(a) $f(x) = x^3 - 3x - 5$.

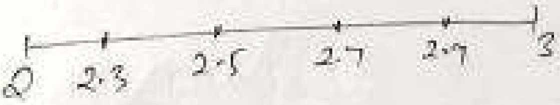
$f(0) = 0^3 - 3\times 0 - 5 = -5 < 0$

$f(1) = 1^3 - 3\times(1) - 5 = -7 < 0$

$f(2) = 2^3 - 3(2) - 5 = 8 - 6 - 5 = -3 < 0$

$f(3) = 3^3 - 3(3) - 5 = 27 - 9 - 5 = 13 > 0$

$\therefore$ Root lies b/w 2 & 3.



$f(2.5) = (2.5)^3 - 3(2.5) - 5 = 3.125 > 0$

$f(2.4) = (2.4)^3 - 3(2.4) - 5 = 1.624 > 0$

$f(2.3) = (2.3)^3 - 3(2.3) - 5 = 0.267 > 0$

$f(2.2) = (2.2)^3 - 2(2.2) - 5 = -0.952 < 0$

$\therefore$ Root lies b/w (2.3) & (2.2).

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Using Newton Raphson method.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Ist St. $i = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.2 - \frac{f(2.2)}{f'(2.2)} \qquad = \left[\begin{array}{l} x^3 - 3x - 5) \\ f' = 3x^2 - 3 = 0 \end{array}\right]$$

$$= (11.52)$$

$$= 2.2 - \frac{0.952}{11.52}$$

$$= 2.2 - 0.08263$$

$$x_1 = 2.11737$$

$$\left[\begin{array}{l} f(2.11737) = \cancel{6.0392}\ -2.8593 \\ f'(2.11737) = 10.4497 \end{array}\right]$$

IInd St. $i = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\cancel{= 2.1173 - \frac{6.0392}{10.4497}} = 2.11737 - \frac{(-2.8593)}{10.4497}$$

$$\cancel{= 2.1173 - 0.5775} \qquad = 2.11737 + 0.17792$$

$$= 2.29529$$

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3rd It^n:-

$i = 2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \quad \begin{cases} f(x_2) = f(2.29529) = 0.20653 \\ f'(x_2) = 12.80506 \end{cases}$$

$$= 2.29529 - \frac{0.20653}{12.80506}$$

$$= 2.29529 - 0.01612$$

$$x_3 = 2.27917$$

4th It^n:-

$i = 3$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \quad \begin{cases} f(x_3) = -0.0019026 \\ f'(x_3) = 12.58384 \end{cases}$$

$$= 2.27917 + \frac{0.0019026}{12.58384}$$

$\cancel{= 2.279274}$

$$x_4 = 2.27932$$

∴ After Compare 3rd & 4th It^n the root of eq. is 2.2793 up to four digits.

Some deductions from Newton Raphson Method.

i) Iterative formula to find $\frac{1}{N}$ is.

$$\boxed{x_{n+1} = x_n(2 - Nx_n).}$$

proof.

Let,

$$x = \frac{1}{N}$$

Take reciprocal

$$\frac{1}{x} = N$$

$$f(x) = \frac{1}{x} - N = 0. \qquad f'(x) = -\frac{1}{x^2}$$

Using Newton Raphson method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

$$= x_n + \frac{\frac{1}{x_n} - N}{\frac{1}{x_n^2}}$$

$$= x_n + \left(\frac{1}{x_n} - N\right) x_n^2$$

$$= x_n + x_n - x_n^2 N$$

$$= 2x_n - x_n^2 N$$

$x_n(2 - Nx_n)$ proof.

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(ii) Iterative formula to find $\sqrt{N}$

$$\boxed{x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]}.$$

proof:

Let,

$$x = \sqrt{N}$$

square both side

$$x^2 = N$$

$$x^2 - N = 0$$

$$f(x) = x^2 - N = 0 \quad f'(x) = 2x.$$

using newton Raphson method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - N}{2x_n}$$

$$= \frac{2x(x_n) - x_n^2 + N}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$$= \frac{x_n^2 + N}{2x_n} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$$ prove

(iii) Iterative formula to find $\frac{1}{\sqrt{N}}$:

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{1}{Nx_n}\right]$$

Proof:-

lets

$$x = \frac{1}{\sqrt{N}}$$

square both side.

$$x^2 = \frac{1}{N}$$

Take reciprocal.

$$\frac{1}{x^2} = N$$

$$\cancel{\boxed{f(x) = \frac{1}{x^2} - N = 0. \quad f'(x) =}}$$

$$f(x) = Nx^2 - 1 = 0 \qquad f'(x) = 2Nx.$$

by using newton raphson method.

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

$$= x_n - \frac{Nx_n^2 - 1}{2Nx_n}$$

$$= \frac{2Nx_n^2 - Nx_n^2 + 1}{2Nx_n} = \frac{Nx_n^2 + 1}{2Nx_n}$$

$$= \frac{1}{2}\left(\frac{N x_n^2}{N x_n} + \frac{1}{N x_n}\right)$$

$$= \frac{1}{2}\left[x_n + \frac{1}{N x_n}\right]. \text{ prove.}$$

iv) Iteration formula to find $K\sqrt{N}$.

$$x_{n+1} = \frac{1}{K}\left[(K-1)x_n + \frac{N}{x_n^{K-1}}\right].$$

Let.

$$x = K\sqrt{N}$$

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Interpolation for unequal Intervals

① Lagrange's Interpolation Method.
② Newton's divided diff. Method.

Lagrange's Interpolation Method

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad - - -$
$y_0 \quad y_1 \quad y_2 \quad y_3$

$$\boxed{y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3}$$

Q Find polynomial $f(x)$ by using Lagrange's Int. method for data

x	0	1	2	5
y	2	3	12	147

& hence obtain $f(3)$

Sol

$$y = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} \times 2 + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} \times 3 + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} \times 12 + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} \times 147$$

$$y = \frac{(x^2-2x-x+2)(x-5)}{(-1)(-2)(-5)} \times 2 + \frac{(x^2-2x)(x-5)}{(1)(-1)(-4)} \times 3 + \frac{(x^2-x)(x-5)}{(2)(1)(-3)} \times 12 + \frac{(x^2-x)(x-2)}{(5)(4)(3)} \times 147$$

$$y = \frac{(x^2-3x+2)(x-5)}{-5} + \frac{3}{4}\left[(x^2-2x)(x-5)\right] - 2\left[(x^2-x)(x-5)\right] + \frac{49}{20}\left[(x^2-x)(x-2)\right]$$

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$$y = \frac{-1}{5}\left[x^3 - 5x^2 - 3x^2 + 15x + 2x - 10\right] + \frac{3}{4}\left[x^3 - 7x^2 + 10x\right]$$

$$- 2\left[x^3 - 5x^2 - x^2 + 5x\right] + \frac{49}{20}\left[x^3 - 2x^2 - x^2 + 2x\right]$$

$$y = \frac{-1}{5}\left[x^3 - 8x^2 + 17x - 10\right] + \frac{3}{4}\left[x^3 - 7x^2 + 10x\right]$$

$$- 2\left[x^3 - 6x^2 + 5x\right] + \frac{49}{20}\left[x^3 - 3x^2 + 2x\right]$$

$$y = \frac{-1}{5}\left[x^3 - 8x^2 + 17x - 10\right] \quad y = \left[\frac{-x^3}{5} + \frac{3}{4}x^3 - 2x^3 + \frac{49}{20}x^3\right]$$

$$+ \left[\frac{8}{5}x^2 - \frac{21}{4}x^2 + 12x^2 - \frac{147}{20}x^2\right]$$

$$+ \left[-\frac{17}{5}x + \frac{15}{2}x - 10x + \frac{49}{10}x\right] + 2$$

$$= \frac{-4x^3 + 15x^3 - 40x^3 + 49x^3}{20} +$$

$$\frac{32x^2 - 105x^2 + 240x^2 - 147x^2}{20} +$$

$$\left[\frac{-34x + 75x - 100x + 49x}{10}\right] + 2$$

$=$

$$f(x) = x^3 + x^2 - x + 2 \quad \text{Ans}$$

$$f(3) = 35$$

$f(9) = 810$

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Q Find $f(9)$ using Lag. for.

x: 5 7 11 13 17

y: 150 392 1452 2366 5202.

Sol

$$y = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$$y = \frac{-128}{1152} \times 150 + \frac{(-256)}{(-480)} \times 392 + 1290.66$$

$$y = -16.6 + 209.066 + 1290.66 + (-788.66) + 115.6$$

$$= 810.066 \text{ ans}$$

Divided diff. table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	150				
		$\frac{392-150}{7-5} = 121$			
7	392		$\frac{265-121}{11-5} = 24$		
		$\frac{1452-392}{11-7} = 265$		$\frac{32-24}{13-5} = 1$	
11	1452		$\frac{457-265}{13-7} = 32$		$\frac{1-1}{17-5} = 0$
		$\frac{2366-1452}{13-11} = 457$		$\frac{42-32}{17-7} = 1$	
13	2366		$\frac{709-457}{17-11} = 42$		
		$\frac{5202-2366}{17-13} = 709$			
17	5202				

Newton's divided diff. formula

$$\boxed{y = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + \dots}$$

$$y = 150 + (9-5)(121) + (9-5)(9-7)(24) + (9-5)(9-7)(9-11)(1)$$

$$= 810$$

Q find $f(8)$ & $f(15) = ?$

x :	4_{x_0}	5_{x_1}	7_{x_2}	10_{x_3}	11_{x_4}	13_{x_5}
y :	48	100	294	900	1210	2028

Sol Using Lagrange's Interpolation method.

~~$y = (x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)(x-x_6)$~~

~~$x = f(8)$~~

$$y = \left[\frac{(8-5)(8-7)(8-10)(8-11)(8-13)}{(4-5)(4-7)(4-10)(4-11)(4-13)} \times 48\right] + \left[\frac{(8-4)(8-7)(8-10)(8-11)(8-13)}{(5-4)(5-7)(5-10)(5-11)(5-13)} \times 100\right]$$

$$+ \left[\frac{(8-4)(8-5)(8-10)(8-11)(8-13)}{(7-4)(7-5)(7-10)(7-11)(7-13)} \times 294\right] + \left[\frac{(8-4)(8-5)(8-7)(8-11)(8-13)}{(10-4)(10-5)(10-7)(10-11)(10-13)} \times 900\right]$$

$$+ \left[\frac{(8-4)(8-5)(8-7)(8-10)(8-13)}{(11-4)(11-5)(11-7)(11-10)(11-13)} \times 1210\right] + \left[\frac{(8-4)(8-5)(8-7)(8-10)(8-11)}{(13-4)(13-5)(13-7)(13-10)(13-11)} \times 2028\right]$$

~~$$= \left[\frac{-90}{-1134} \times 48\right] + \left[\frac{-120}{480} \times 100\right] + \left[\frac{-360}{-432} \times 1210\right] + \left[\frac{72}{2592} \times 2028\right]$$~~

~~$$= 3.809 - 25 - 432.142 + 56.33$$~~

$$= \left[\frac{+90}{+1134} \times 48\right] + \left[\frac{-120}{480} \times 100\right] + \left[\frac{+360}{+432} \times 294\right] + \left[\frac{180}{270} \times 900\right]$$

$$+ \left[\frac{120}{-336} \times 1210\right] + \left[\frac{72}{2592} \times 2028\right]$$

$$= 3.809 - 25 + 245 + 600 - 432.142 + 56.33$$

$$= 447.997 \simeq 448$$

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f(15)

$$y = \left[\frac{(15-5)(15-7)(15-10)(15-11)(15-13)}{(4-5)(4-7)(4-10)(4-11)(4-13)} \times 48\right] + \left[\frac{(15-4)(15-7)(15-10)(15-11)(15-13)}{(5-4)(5-7)(5-10)(5-11)(5-13)} \times 100\right]$$

$$+ \left[\frac{(15-4)(15-5)(15-10)(15-11)(15-13)}{(7-4)(7-5)(7-10)(7-11)(7-13)} \times 294\right] +$$

$$\left[\frac{(15-4)(15-5)(15-7)(15-11)(15-13)}{(10-4)(10-5)(10-7)(10-11)(10-13)} \times 900\right] + \left[\frac{(15-4)(15-5)(15-7)(15-10)(15-13)}{(11-4)(11-5)(11-7)(11-10)(11-13)} \times 1210\right]$$

$$+ \left[\frac{(15-4)(15-5)(15-7)(15-10)(15-11)}{(13-4)(13-5)(13-7)(13-10)(13-11)} \times 2028\right]$$

$$y = \left[\frac{3200}{-1134} \times 48\right] + \left[\frac{3520}{480} \times 100\right] + \left[\frac{4400}{-432} \times 294\right] + \left[\frac{7040}{270} \times 900\right]$$

$$+ \left[\frac{8800}{-336} \times 1210\right] + \left[\frac{17600}{2592} \times 2028\right]$$

$$y = -135.449 + 733.33 - 2994.44 + 23466.66 + (-31690.47) + 13770.37$$

$$= 3150.001 \Rightarrow 3150$$

By using Lagrange's Interpolation method we get

$$\left.\begin{aligned} \therefore\ f(8) &= 448 \\ \& \ f(15) &= 3150 \end{aligned}\right\} \text{ans.}$$

By using Divided diff. table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
4	48	$\frac{100-48}{5-4} = 52$	$\frac{97-52}{7-4} = 15$	$\frac{21-15}{10-4} = 1$	$\frac{1-1}{11-4} = 0$	0
5	100	$\frac{294-100}{7-5} = 97$	$\frac{202-97}{10-5} = 21$	$\frac{27-21}{11-5} = 1$	$\frac{1-1}{13-5} = 0$	
7	294	$\frac{900-294}{10-7} = 202$	$\frac{310-202}{11-7} = 27$	$\frac{33-27}{13-7} = 1$		
10	900	$\frac{1210-900}{11-10} = 310$	$\frac{409-310}{13-10} = 33$			
11	1210	$\frac{2028-1210}{13-11} = 409$				
13	2028					

$$y = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0$$
$$+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)\Delta^4 y_0 + (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)\Delta^5 y_0$$

$f(8)$

$$y = 48 + (8-4)(52) + (8-4)(8-5)(15) + (8-4)(8-5)(8-7)(1) +$$
$$(8-4)(8-5)(8-7)(8-10)(0) + (8-4)(8-5)(8-7)(8-10)(8-11)(0)$$

$$y = 48 + 208 + 180 + 12 + 0 + 0 = 448$$

$f(15)$

$$y = 48 + (15-4)(52) + (15-4)(15-5)(15) + (15-4)(15-5)(15-7)(1)$$
$$+ 0 + 0$$

$$y = 48 + 572 + 1650 + 880 = 3150 \text{ ans}$$

Example 10.20. *Using Milne's method find $y(4.4)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$; $y(4.4) = 1.0187$.* (*Anna, B.E., 2007*)

Sol. We have $y' = (2 - y^2)/5x = f(x)$ [say]

Then the starting values of the Milne's method are

$$x_0 = 4, \quad y_0 = 1, \qquad f_0 = \frac{2 - 1^2}{5 \times 4} = 0.05$$

$$x_1 = 4.1,\ y_1 = 1.0049, \qquad f_1 = 0.0485$$
$$x_2 = 4.2,\ y_2 = 1.0097, \qquad f_2 = 0.0467$$
$$x_3 = 4.3,\ y_3 = 1.0143, \qquad f_3 = 0.0452$$
$$x_4 = 4.4,\ y_4 = 1.0187, \qquad f_4 = 0.0437$$

Since y_5 is required, we use the *predictor*

$$y_5^{(p)} = y_1 + \frac{4h}{3}\ (2f_2 - f_3 + 2f'_4) \qquad (h = 0.1)$$

$$x = 4.5, \qquad y_5^{(p)} = 1.0049 + \frac{4(0.1)}{3}\ (2 \times 2.0467 - 0.0452 + 2 \times 0.0437) = 1.023$$

$$f_5 = \frac{2 - {y_5}^2}{5x_5} = \frac{2 - (1.023)^2}{5 \times 4.5} = 0.0424$$

Now using the *corrector* $y_5^{(c)} = y_3 + \dfrac{h}{3}\ (f_3 + 4f_4 + f_5)$, we get

$$y_5^{(c)} = 1.0143 + \frac{0.1}{3}\ (0.0452 + 4 \times 0.0437 + 0.0424) = 1.023.$$

Hence $\qquad y(4.5) = 1.023.$

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Example 10.21. *Given $y' = x(x^2 + y^2)\, e^{-x}$, $y(0) = 1$, find y at $x = 0.1$, 0.2 and 0.3 by Taylor's series method and compute $y(0.4)$ by Milne's method.* (Anna, B.E., 2007)

Sol. Given $y(0) = 1$ and $h = 0.1$

We have $y'(x) = x\,(x^2 + y^2)\, e^{-x}$; $\quad y'(0) = 0$

$$\therefore \quad y''(x) = [(x^3 + xy^2)(-e^{-x}) + (3x^2 + y^2 + x(2y)\,y']\, e^{-x}$$

$$= e^{-x}\,[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy'] \;; \qquad y''(0) = 1$$

$$y'''(x) = -e^{-x}\,[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy' + 3x^2 + y^2 + 2xyy' - 6x - 2yy' - 2xy'^2 - 2xyy']$$

$$y'''(0) = -2$$

Substituting these values in the Taylor's series,

$$y(x) = y(0) + \frac{x}{1!}\, y'(0) + \frac{x^2}{2!}\, y''(0) + \frac{x^3}{3!}\, y'''(0) + \ldots\ldots$$

$$y(0.1) = 1 + (0.1)(0) + \frac{1}{2}\,(0.1)^2\,(1) + \frac{1}{6}\,(0.1)^3\,(-2) + \ldots\ldots$$

$$= 1 + 0.005 - 0.0003 = 1.0047 \text{ i.e., } 1.005$$

Now taking $x = 0.1$, $y(0.1) = 1.005$, $h = 0.1$

$$y'(0.1) = 0.092,\; y''(0.1) = 0.849 \;;\; y'''(0.1) = -1.247$$

Substituting these values in the Taylor's series about $x = 0.1$,

$$y(0.2) = y(0.1) + \frac{0.1}{1!}\, y'(0.1) + \frac{(0.1)^2}{2!}\, y''(0.1) + \frac{(0.1)^3}{3!}\, y'''(0.1) + \ldots\ldots$$

$$= 1.005 + (0.1)(0.092) + \frac{(0.1)^2}{2}\,(0.849) + \frac{(0.1)^3}{6}\,(-1247) + \ldots\ldots$$

$$= 1.018$$

Now taking $x = 0.2$, $y(0.2) = 1.018$, $h = 0.1$

$$y'(0.2) = 0.176,\; y''(0.2) = 0.77,\; y'''(0.2) = 0.819$$

Substituting these values in the Taylors series

$$y(0.3) = y(0.2) + \frac{0.1}{1!}\, y''(0.2) + \frac{(0.1)^2}{2!}\, y''(0.2) + \frac{(0.1)^3}{3!}\, y'''(0.2) + \ldots\ldots$$

$$= 1.018 + 0.0176 + 0.0039 + 0.0001$$

$$= 1.04$$

Thus the starting values of the Milne's method with $h = 0.1$ are

$x_0 = 0.0$	$y_0 = 1$	$f_0 = y'_0 = 0$
$x_1 = 0.1$	$y_1 = 1.005$	$f_1 = 0.092$
$x_2 = 0.2$	$y_2 = 1.018$	$f_2 = 0.176$
$x_3 = 0.3$	$y_3 = 1.04$	$f_3 = 0.26$

Using the predictor, $y_4^{(p)} = y_0 + \dfrac{4h}{3}\,(2f_1 - f_2 + 2f_3)$

$$= 1 + \frac{4(0.1)}{3}\,[2(0.092) - 0.176 + 2(0.26)]$$

$$= 1.09.$$

$\therefore \quad x = 0.4 \qquad y_4^{(p)} = 1.09 \qquad f_4 = y'(0.4) = 0.362$

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Using the corrector, $y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$

$$\therefore \quad y_4^{(c)} \doteq 0.018 + \frac{0.1}{3}(0.176 + 4(0.26) + 0.362) = 1.071$$

Hence $\quad y(0.4) = 1.071.$

Example 10.22. *Using Runge-Kutta method of order 4, find y for x = 0.1, 0.2, 0.3 given that $dy/dx = xy + y^2$, $y(0) = 1$. Continue the solution at x = 0.4 using Milne's method.*

(S.V.T.U., B. Tech., 2007)

Sol. We have $\quad f(x, y) = xy + y^2.$

To find y(0.1) :

Here $x_0 = 0$, $y_0 = 1$, $h = 0.1$.

$$\therefore \quad k_1 = h\,f(x_0, y_0) = (0.1)f(0, 1) \qquad = 0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1)\,f(0.05, 1.05) \qquad = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1)\,f(0.05, 1.0577) \qquad = 0.1172$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3) = (0.1)\,f(0.1, 1.1172) \qquad = 0.13598$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.231 + 0.2343 + 0.13598) \qquad = 0.11687$$

Thus $\quad y(0.1) = y_1 = y_0 + k = 1.1169.$

To find y(0.2) :

Here $x_1 = 0.1$, $y_1 = 1.1169$, $h = 0.1$.

$$k_1 = h\,f(x_1, y_1) = (0.1)\,f(0.1, 1.1169) \qquad = 0.1359$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1)\,f(0.15, 1.1848) \qquad = 0.1581$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1)\,f(0.15, 1.1959) \qquad = 0.1609$$

$$k_4 = h\,f(x_1 + h, y_1 + k_3) = (0.1)\,f(0.2, 1.2778) \qquad = 0.1888$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.1605$$

Thus $\quad y(0.2) = y_2 = y_1 + k = 1.2773.$

To find y(0.3) :

Here $x_2 = 0.2$, $y_2 = 1.2773$, $h = 0.1$.

$$k_1 = h\,f(x_2, y_2) = (0.1)\,f(0.2, 1.2773) \qquad = 0.1887$$

$$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1)\,f(0.25, 1.3716) \qquad = 0.2224$$

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$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1)\,f(0.25,\ 1.3885) \qquad = 0.2275$$

$$k_4 = h\,f(x_2 + h, y_2 + k_3) = (0.1)\,f(0.3,\ 1.5048) \qquad = 0.2716$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.2267$$

Thus $y(0.3) = y_3 = y_2 + k = 1.504$

Now the starting values for the Milne's method are :

$x_0 = 0.0$	$y_0 = 1.0000$	$f_0 = 1.0000$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$f_2 = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$f_3 = 2.7132$

Using the *predictor*,

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$x_4 = 0.4 \qquad y_4^{(p)} = 1.8344 \qquad f_4 = 4.0988$

and the *corrector*,

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \text{ yields}$$

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.098]$$

$$= 1.8386 \qquad f_4 = 4.1159.$$

Again using the *corrector*,

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1159]$$

$$= 1.8391 \qquad f_4 = 4.1182$$

Again using the *corrector*

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1182]$$

$$= 1.8392 \text{ which is same as } (i).$$

Hence $y(0.4) = 1.8392$.

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Euler's Modified Method

ODE : $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$

$y(x_0)$ ~~$\checkmark$~~

$y(\text{some point}) = ?$

Formula used:

$y_{n+1} = y_n + h\, f(x_n, y_n)$ → Euler's Formula

modification number in (y+1)

$y_{n+1}^{(m)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(m-1)})\right]$ → Euler's Modified Formula

$x_n = x_{n-1} + h$

$x_1 = x_0 + h$, $x_2 = x_1 + h$, $x_3 = x_2 + h$ - - - ---

Q If $\frac{dy}{dx} = x + y^2$ and $y = 1$ at $x = 0$, find an approximate value of y at $x = 0.2$ by improved Euler's modified Method (taking $h = 0.1$)

Sol Given → $h = 0.1$, $x_0 = 0$, $y_0 = y(x_0) = 1$, $f(x,y) = x + y^2$

To find → $y(0.2) = ?$ $x_1 = x_0 + h = 0 + 1 = 1$, $x_2 = x_1 + h = 0.2$

By Euler's Modified Method

$y_{n+1} = y_n + h\, f(x_n, y_n)$ —— ①

$y_{n+1}^{(m)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(m-1)})\right]$ —— ②

Step 1 put $(n=0)$ into ①

$y_1 = y(x_1) = y(0.1) = y_0 + h f(x_0, y_0)$

$= 1 + 0.1\left[x_0 + y_0^2\right]$

$= 1 + 0.1\left[0 + 1^2\right]$

$\boxed{y_1 = 1.1}$

Put $(n=0)$ into ② (first Modification in y)
$m=1$

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$= 1 + \frac{0.1}{2}\left[x_0 + y_0^2 + x_1 + y_1^{(0)2}\right]$

$= 1 + \frac{0.1}{2}\left[0 + 1^2 + 0.1 + (1.1)^2\right]$

$\boxed{y_1^{(1)} = 1.1155}$

$$\boxed{y(0.1) = 1.1155}$$

Step-2 put $n=1$ into (1)

$$y_2 = y(0.2) = y_1 + h f(x_1, y_1)$$
$$= 1.1155 + 0.1 [x_1 + y_1^2]$$
$$= 1.1155 + 0.1 [0.1 + 1.1155^2]$$
$$\boxed{y_2 = y(0.2) = 1.2499}$$

put $n=1$, $m=1$ into (2) 1st Modification in y.

$$y_2^{(1)} = y_1 + \frac{h}{2} [F(x_p, y_p) + F(x_2, y_2^{(0)})]$$
$$y_2^{(1)} = 1.1155 + \frac{0.1}{2} [x_1 + y_1^2 + x_2 + y_2^{(0)2}]$$
$$= 1.1155 + \frac{0.1}{2} [0.1 + 1.1155^2 + 0.2 + 1.2499^2]$$
$$\boxed{y_2^{(1)} = 1.2708} \quad \text{or} \quad \boxed{y_2 = y(0.2) = 1.2708}$$

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Example 10.20. *Using Milne's method find y(4.4) given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$; $y(4.4) = 1.0187$.* (Anna, B.E., 2007)

Sol. We have $y' = (2 - y^2)/5x = f(x)$ [say]

Then the starting values of the Milne's method are

$$x_0 = 4, \quad y_0 = 1, \qquad f_0 = \frac{2 - 1^2}{5 \times 4} = 0.05$$

$$x_1 = 4.1, \; y_1 = 1.0049, \qquad f_1 = 0.0485$$
$$x_2 = 4.2, \; y_2 = 1.0097, \qquad f_2 = 0.0467$$
$$x_3 = 4.3, \; y_3 = 1.0143, \qquad f_3 = 0.0452$$
$$x_4 = 4.4, \; y_4 = 1.0187, \qquad f_4 = 0.0437$$

Since y_5 is required, we use the *predictor*

$$y_5^{(p)} = y_1 + \frac{4h}{3}\,(2f_2 - f_3 + 2f'_4) \qquad (h = 0.1)$$

$$x = 4.5, \qquad y_5^{(p)} = 1.0049 + \frac{4(0.1)}{3}\,(2 \times 2.0467 - 0.0452 + 2 \times 0.0437) = 1.023$$

$$f_5 = \frac{2 - {y_5}^2}{5x_5} = \frac{2 - (1.023)^2}{5 \times 4.5} = 0.0424$$

Now using the *corrector* $y_5^{(c)} = y_3 + \dfrac{h}{3}\,(f_3 + 4f_4 + f_5)$, we get

$$y_5^{(c)} = 1.0143 + \frac{0.1}{3}\,(0.0452 + 4 \times 0.0437 + 0.0424) = 1.023.$$

Hence $\qquad y(4.5) = 1.023.$

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Example 10.21. *Given $y' = x(x^2 + y^2)\, e^{-x}$, $y(0) = 1$, find y at $x = 0.1$, 0.2 and 0.3 by Taylor's series method and compute $y(0.4)$ by Milne's method.* (Anna, B.E., 2007)

Sol. Given $\quad y(0) = 1$ and $h = 0.1$

We have $\quad y'(x) = x\,(x^2 + y^2)\, e^{-x}$; $\quad y'(0) = 0$

$\therefore \quad y''(x) = [(x^3 + xy^2)(-e^{-x}) + (3x^2 + y^2 + x(2y)\, y']\, e^{-x}$

$= e^{-x}\,[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy']$; $\quad y''(0) = 1$

$y'''(x) = -e^{-x}\,[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy' + 3x^2 + y^2 + 2xyy' - 6x - 2yy' - 2xy'^2 - 2xyy']$

$y'''(0) = -2$

Substituting these values in the Taylor's series,

$$y(x) = y(0) + \frac{x}{1!}\, y'(0) + \frac{x^2}{2!}\, y''(0) + \frac{x^3}{3!}\, y'''(0) + \ldots\ldots$$

$$y(0.1) = 1 + (0.1)(0) + \frac{1}{2}(0.1)^2(1) + \frac{1}{6}(0.1)^3(-2) + \ldots\ldots$$

$$= 1 + 0.005 - 0.0003 = 1.0047 \text{ i.e., } 1.005$$

Now taking $\quad x = 0.1,\ y(0.1) = 1.005,\ h = 0.1$

$y'(0.1) = 0.092,\ y''(0.1) = 0.849$; $y'''(0.1) = -1.247$

Substituting these values in the Taylor's series about $x = 0.1$,

$$y(0.2) = y(0.1) + \frac{0.1}{1!}\, y'(0.1) + \frac{(0.1)^2}{2!}\, y''(0.1) + \frac{(0.1)^3}{3!}\, y'''(0.1) + \ldots\ldots$$

$$= 1.005 + (0.1)(0.092) + \frac{(0.1)^2}{2}(0.849) + \frac{(0.1)^3}{6}(-1247) + \ldots\ldots$$

$$= 1.018$$

Now taking $\quad x = 0.2,\ y(0.2) = 1.018,\ h = 0.1$

$y'(0.2) = 0.176,\ y''(0.2) = 0.77,\ y'''(0.2) = 0.819$

Substituting these values in the Taylors series

$$y(0.3) = y(0.2) + \frac{0.1}{1!}\, y''(0.2) + \frac{(0.1)^2}{2!}\, y''(0.2) + \frac{(0.1)^3}{3!}\, y'''(0.2) + \ldots\ldots$$

$$= 1.018 + 0.0176 + 0.0039 + 0.0001$$

$$= 1.04$$

Thus the starting values of the Milne's method with $h = 0.1$ are

$x_0 = 0.0$	$y_0 = 1$	$f_0 = y'_0 = 0$
$x_1 = 0.1$	$y_1 = 1.005$	$f_1 = 0.092$
$x_2 = 0.2$	$y_2 = 1.018$	$f_2 = 0.176$
$x_3 = 0.3$	$y_3 = 1.04$	$f_3 = 0.26$

Using the predictor, $y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

$$= 1 + \frac{4(0.1)}{3}[2(0.092) - 0.176 + 2(0.26)]$$

$$= 1.09.$$

$\therefore \quad x = 0.4 \quad y_4^{(p)} = 1.09 \quad f_4 = y'(0.4) = 0.362$

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Using the corrector, $y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$

$$\therefore \qquad y_4^{(c)} \doteq 0.018 + \frac{0.1}{3}(0.176 + 4(0.26) + 0.362) = 1.071$$

Hence $\qquad y(0.4) = 1.071.$

Example 10.22. *Using Runge-Kutta method of order 4, find y for x = 0.1, 0.2, 0.3 given that $dy/dx = xy + y^2$, $y(0) = 1$. Continue the solution at x = 0.4 using Milne's method.*

(S.V.T.U., B. Tech., 2007)

Sol. We have $\qquad f(x, y) = xy + y^2.$

To find y(0.1) :

Here $x_0 = 0$, $y_0 = 1$, $h = 0.1$.

$$\therefore \qquad k_1 = h\,f(x_0, y_0) = (0.1) f(0, 1) \qquad = 0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1)\,f(0.05, 1.05) \qquad = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1)\,f(0.05, 1.0577) \qquad = 0.1172$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3) = (0.1)\,f(0.1, 1.1172) \qquad = 0.13598$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.231 + 0.2343 + 0.13598) \qquad = 0.11687$$

Thus $\qquad y(0.1) = y_1 = y_0 + k = 1.1169.$

To find y(0.2) :

Here $x_1 = 0.1$, $y_1 = 1.1169$, $h = 0.1$.

$$k_1 = h\,f(x_1, y_1) = (0.1)\,f(0.1, 1.1169) \qquad = 0.1359$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1)\,f(0.15, 1.1848) \qquad = 0.1581$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1)\,f(0.15, 1.1959) \qquad = 0.1609$$

$$k_4 = h\,f(x_1 + h, y_1 + k_3) = (0.1)\,f(0.2, 1.2778) \qquad = 0.1888$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.1605$$

Thus $\quad y(0.2) = y_2 = y_1 + k = 1.2773.$

To find y(0.3) :

Here $x_2 = 0.2$, $y_2 = 1.2773$, $h = 0.1$.

$$k_1 = h\,f(x_2, y_2) = (0.1)\,f(0.2, 1.2773) \qquad = 0.1887$$

$$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1)\,f(0.25, 1.3716) \qquad = 0.2224$$

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$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1)\, f(0.25,\ 1.3885) \qquad = 0.2275$$

$$k_4 = h\, f(x_2 + h, y_2 + k_3) = (0.1)\, f(0.3,\ 1.5048) \qquad = 0.2716$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.2267$$

Thus $y(0.3) = y_3 = y_2 + k = 1.504$

Now the starting values for the Milne's method are :

$x_0 = 0.0$	$y_0 = 1.0000$	$f_0 = 1.0000$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$f_2 = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$f_3 = 2.7132$

Using the *predictor*,

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$x_4 = 0.4 \qquad y_4^{(p)} = 1.8344 \qquad\qquad f_4 = 4.0988$

and the *corrector*,

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \text{ yields}$$

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.098]$$

$$= 1.8386 \qquad\qquad f_4 = 4.1159.$$

Again using the *corrector*,

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1159]$$

$$= 1.8391 \qquad\qquad f_4 = 4.1182$$

Again using the *corrector*

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1182]$$

$$= 1.8392 \text{ which is same as } (i).$$

Hence $y(0.4) = 1.8392$.

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Euler's Modified Method

ODE : $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$

$y(x_0)$ ~~x~~

$y(\text{some point}) = ?$

Formula used:

$y_{n+1} = y_n + h\, f(x_n, y_n)$ → Euler's Formula

modification number in $(y+1)$

$y_{n+1}^{(m)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(m-1)})\right]$ → Euler's Modified Formula

$x_n = x_{n-1} + h$

$x_1 = x_0 + h,\quad x_2 = x_1 + h,\quad x_3 = x_2 + h \ldots$

Q If $\frac{dy}{dx} = x + y^2$ and $y = 1$ at $x = 0$, find an approximate value of y at $x = 0.2$ by improved Euler's modified Method (taking $h = 0.1$)

Sol Given → $h = 0.1,\ x_0 = 0,\ y_0 = y(x_0) = 1,\ f(x,y) = x + y^2$

To find → $y(0.2) = ?$ $\qquad x_1 = x_0 + h = 0 + 1 = 1,\ x_2 = x_1 + h = 0.2$

By Euler's Modified Method

$$y_{n+1} = y_n + h\, f(x_n, y_n) \quad \text{--- (1)}$$

$$y_{n+1}^{(m)} = y_n + \frac{h}{2}\left[f(x_n, y_m) + f(x_{n+1}, y_{n+1}^{(m-1)})\right] \quad \text{--- (2)}$$

Step 1 put $(n=0)$ into (1)

$$y_1 = y(x_1) = y(0.1) = y_0 + h f(x_0, y_0)$$
$$= 1 + 0.1\left[x_0 + y_0^2\right]$$
$$= 1 + 0.1\left[0 + 1^2\right]$$
$$\boxed{y_1 = 1.1}$$

Put $(n=0)$ into (2) (first modification in y)

$m = 1$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$
$$= 1 + \frac{0.1}{2}\left[x_0 + y_0^2 + x_1 + y_1^{(0)2}\right]$$
$$= 1 + \frac{0.1}{2}\left[0 + 1^2 + 0.1 + (1.1)^2\right]$$
$$\boxed{y_1^{(1)} = 1.1155}$$

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$$\boxed{y(0.1) = 1.1155}$$

Step-2 put $n=1$ into (1)

$$y_2 = y(0.2) = y_1 + h f(x_1, y_1)$$

$$= 1.1155 + 0.1 [x_1 + y_1^2]$$

$$= 1.1155 + 0.1 [0.1 + 1.1155^2]$$

$$\boxed{y_2 = y(0.2) = 1.2499}$$

put $n=1$, $m=1$ into (2) 1st Modification in y.

$$y_2^{(1)} = y_1 + \frac{h}{2} [F(x_p, y_p) + F(x_2, y_2^{(0)})]$$

$$y_2^{(1)} = 1.1155 + \frac{0.1}{2} [x_1 + y_1^2 + x_2 + y_2^{(0)2}]$$

$$= 1.1155 + \frac{0.1}{2} [0.1 + 1.1155^2 + 0.2 + 1.2499^2]$$

$$\boxed{y_2^{(1)} = 1.2708} \quad \text{or} \quad \boxed{y_2 = y(0.2) = 1.2708}$$

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almost equal.

Example 1. *Given that* $\frac{dy}{dx} = y - x$ *where* $y(0) = 2$*, find* $y(0.1)$ *and* $y(0.2)$ *by Euler's method upto two decimal places.* (P.T.U. Dec. 200

Sol. The given initial value problem is

$$\frac{dy}{dx} = y - x,\ y(0) = 2.$$

$$\therefore \qquad f(x, y) = y - x,\ x_0 = 0,\ y_0 = 2$$

take $h = 0.1$

We are to find the value of y at $x = 0.1$, $x = 0.2$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

By Euler's formula

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = 2 + (0.1) f(0, 2)$$

$$= 2 + (0.1)(2 - 0) = 2 + 0.2 = 2.2$$

Now,

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

By Euler's Method

$$y_2 = y_1 + hf(x_1, y_1) = 2.2 + (0.1) f(0.1, 2.2)$$

$$= 2.2 + (0.1)(2.2 - 0.1) = 2.2 + 0.21 = 2.41$$

$$y_2 = 2.41$$

Finite Diff Method :

Replace
$y \to y_i \; , \; x \to x_i$

$$y'' = \frac{y_{i+1} - 2y_i + y_{-1}}{h^2}$$

$$y''' = \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2h^3}$$

Example 10.20. *Using Milne's method find y(4.4) given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$; $y(4.4) = 1.0187$.* (Anna, B.E., 2007)

Sol. We have $y' = (2 - y^2)/5x = f(x)$ [say]

Then the starting values of the Milne's method are

$$x_0 = 4, \quad y_0 = 1, \qquad f_0 = \frac{2 - 1^2}{5 \times 4} = 0.05$$

$$x_1 = 4.1, \ y_1 = 1.0049, \qquad f_1 = 0.0485$$
$$x_2 = 4.2, \ y_2 = 1.0097, \qquad f_2 = 0.0467$$
$$x_3 = 4.3, \ y_3 = 1.0143, \qquad f_3 = 0.0452$$
$$x_4 = 4.4, \ y_4 = 1.0187, \qquad f_4 = 0.0437$$

Since y_5 is required, we use the *predictor*

$$y_5^{(p)} = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f'_4) \qquad (h = 0.1)$$

$$x = 4.5, \quad y_5^{(p)} = 1.0049 + \frac{4(0.1)}{3}(2 \times 2.0467 - 0.0452 + 2 \times 0.0437) = 1.023$$

$$f_5 = \frac{2 - y_5^{\,2}}{5x_5} = \frac{2 - (1.023)^2}{5 \times 4.5} = 0.0424$$

Now using the *corrector* $y_5^{(c)} = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5)$, we get

$$y_5^{(c)} = 1.0143 + \frac{0.1}{3}(0.0452 + 4 \times 0.0437 + 0.0424) = 1.023.$$

Hence $\quad y(4.5) = 1.023.$

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Euler's Modified Method

ODE : $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$

$y(x_0)$ $\cancel{?}$

$y(\text{some point}) = ?$

Formula used:

modification number in $(y+1)$

$y_{n+1} = y_n + h\, f(x_n, y_n)$ → Euler's Formula

$y_{n+1}^{(m)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(m-1)})\right]$ → Euler's Modified Formula

$x_n = x_{n-1} + h$

$x_1 = x_0 + h,\quad x_2 = x_1 + h,\quad x_3 = x_2 + h$ - - - —

Q If $\frac{dy}{dx} = x + y^2$ and $y = 1$ at $x = 0$, find an approximate value of y at $x = 0.2$ by improved Euler's modified Method (taking $h = 0.1$)

Sol Given → $h = 0.1$, $x_0 = 0$, $y_0 = y(x_0) = 1$, $f(x,y) = x + y^2$

To find → $y(0.2) = ?$ $\quad x_1 = x_0 + h = 0+1 = 1$, $x_2 = x_1 + h = 0.2$

By Euler's Modified Method

$$y_{n+1} = y_n + h\, f(x_n, y_n) \quad \text{—— (1)}$$

$$y_{n+1}^{(m)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(m-1)})\right] \quad \text{—— (2)}$$

Step 1 put $(n=0)$ into (1)

$$y_1 = y(x_1) = y(0.1) = y_0 + h f(x_0, y_0)$$
$$= 1 + 0.1\left[x_0 + y_0^2\right]$$
$$= 1 + 0.1\left[0 + 1^2\right]$$
$$\boxed{y_1 = 1.1}$$

Put $(n=0)$ into (2) (first modification in y)
$m=1$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$
$$= 1 + \frac{0.1}{2}\left[x_0 + y_0^2 + x_1 + y_1^{(0)2}\right]$$
$$= 1 + \frac{0.1}{2}\left[0 + 1^2 + 0.1 + (1.1)^2\right]$$
$$\boxed{y_1^{(1)} = 1.1155}$$

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$$\boxed{y(0.1) = 1.1155}$$

Step-2 put $n=1$ into (1)

$$y_2 = y(0.2) = y_1 + h f(x_1, y_1)$$
$$= 1.1155 + 0.1 [x_1 + y_1^2]$$
$$= 1.1155 + 0.1 [0.1 + 1.1155^2]$$
$$\boxed{y_2 = y(0.2) = 1.2499}$$

put $n=1$, $m=1$ into (2) 1st Modification in y.

$$y_2^{(1)} = y_1 + \frac{h}{2} [F(x_p, y_p) + F(x_2, y_2^{(0)})]$$
$$y_2^{(1)} = 1.1155 + \frac{0.1}{2} [x_1 + y_1^2 + x_2 + y_2^{(0)2}]$$
$$= 1.1155 + \frac{0.1}{2} [0.1 + 1.1155^2 + 0.2 + 1.2499^2]$$
$$\boxed{y_2^{(1)} = 1.2708} \quad \text{or} \quad \boxed{y_2 = y(0.2) = 1.2708}$$

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almost equal.

Example 1. *Given that* $\frac{dy}{dx} = y - x$ *where* $y(0) = 2$, *find* $y(0.1)$ *and* $y(0.2)$ *by Euler's method upto two decimal places.* (P.T.U. Dec. 200

Sol. The given initial value problem is

$$\frac{dy}{dx} = y - x, \; y(0) = 2.$$

$$\therefore \quad f(x, y) = y - x, \; x_0 = 0, \; y_0 = 2$$

take $h = 0.1$

We are to find the value of y at $x = 0.1$, $x = 0.2$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

By Euler's formula

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = 2 + (0.1) f(0, 2)$$

$$= 2 + (0.1)(2 - 0) = 2 + 0.2 = 2.2$$

Now,

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

By Euler's Method

$$y_2 = y_1 + hf(x_1, y_1) = 2.2 + (0.1) f(0.1, 2.2)$$

$$= 2.2 + (0.1)(2.2 - 0.1) = 2.2 + 0.21 = 2.41$$

$$y_2 = 2.41$$

UNIT 2

FINITE DIFFERENCE TECHNIQUE

(Initial and boundary value problems of ordinary and partial differential equations)

2.1 INTRODUCTION

In Engineering Mathematics we come across certain differential equations which cannot be solved by exact methods or if the solution exists, the functions of dependent variable needs difficults calculations. In such cases, the numerical methods are used to find approximate numerical values of the solution.

Initial value Problem : An ordinary differential equation of order n is of the form

$$f(x, y, y', y'', ..., y^n) = 0$$

The general solution of above equation is a relation between x, y and n arbitrary constants say

$$g(x, y, c_1, c_2, ..., c_n) = 0$$

where $c_1, c_2, ..., c_n$ are n arbitrary constants.

If certain fixed values are given to these constants then such solution is called particular solution. To obtain a particular solution we need n conditions to find the values of constants. If all the n conditions are given at the same point then the given differential equation along with the n conditions is called initial value problem.

If the n conditions are given at different points then the given differential equation along with n conditions is called boundary value problem.

First of all, we shall discuss the methods to solve initial value problems of first order.

2.2 EULER'S METHOD

Consider initial value problem

$$\frac{dy}{dx} = f(x, y), \; y(x_0) = y_0$$

suppose we want to calculate y_n at $x = x_n$.

Divide the internal $[x_0, x_n]$ into n. Equal portions of length h.

Thus we get division points $x_0, x_1, x_2, ..., x_n$ such that

$$x_i = x_0 + ih, \; i = 1, 2, 3, ... n.$$

integrating (1) w.r.t. x in $[x_{i-1}, x_i]$ we get

$$\int_{y_{i-1}}^{y_i} dy = \int_{x_{i-1}}^{x_i} f(x, y)\,dx$$

$$y_i - y_{i-1} = \int_{x_{i-1}}^{x_i} f(x, y)\,dx$$

$$y_i = y_{i-1} + \int_{x_{i-1}}^{x_i} f(x, y)\,dx \qquad ...(2)$$

Let $f(x_{i-1}, y_{i-1})$ is the approximate value of $f(x, y)$ in internal $[x_{i-1}, x_i]$.

$\therefore$ Equation (2) becomes

$$y_i = y_{i-1} + f(x_{i-1}, y_{i-1}) \int_{x_{i-1}}^{x_i} dx$$

$$y_i = y_{i-1} + f(x_{i-1}, y_{i-1})(x_i - x_{i-1})$$

$$y_i = y_{i-1} + hf(x_{i-1}, y_{i-1})$$

Above eqn. is called Euler's formula.

Note : Euler's Method is the simplest method so far. But this method is very slow and to attain desired accuracy we shall take smaller value of h. Also this method is not suitable for large range of x as the solution so obtained is far different from original value. Hence this method is not useful practically, a better form of it called modified Euler's method is used which is more accurate.

2.3 MODIFIED EULER'S METHOD

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \qquad ...(1)$$

From Euler's method ...(2)

$$y_i = y_{i-1} + hf(x_{i-1}, y_{i-1})$$

Let y_i obtained in eqn. (2) be the initial value

i.e. $$y_i^{(0)} = y_{i-1} + hf(x_{i-1}, y_{i-1})$$

using Trapezoidal Rule in $[x_{i-1}, x_i]$.

We get

$$y_i = y_{i-1} + \frac{h}{2}[f(x_{i-1}, y_{i-1}) + f(x_i, y_i)]$$

Replacing $f(x_i, y_i)$ by $f(x_i, y_i^{(0)})$, we get first approximation

$$y_i^{(1)} = y_{i-1} + \frac{h}{2}[f(x_{i-1}, y_{i-1}) + f(x_i, y_i^{(0)})]$$

Continuing in this way, we get iterative formula

$$y_i^{(n)} = y_{i-1} + \frac{h}{2}[f(x_{i-1}, y_{i-1}) + f(x_i, y_i^{(n-1)})]$$

where $y_i^{(n)}$ is the nth approximation we stop interations where two consecutive approximation are almost equal.

Example 1. *Given that* $\frac{dy}{dx} = y - x$ *where* $y(0) = 2$, *find* $y(0.1)$ *and* $y(0.2)$ *by Euler's method upto two decimal places.* (P.T.U. Dec. 200

Sol. The given initial value problem is

$$\frac{dy}{dx} = y - x, \; y(0) = 2.$$

$$\therefore \quad f(x, y) = y - x, \; x_0 = 0, \; y_0 = 2$$

take $h = 0.1$

We are to find the value of y at $x = 0.1$, $x = 0.2$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

By Euler's formula

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = 2 + (0.1) f(0, 2)$$

$$= 2 + (0.1)(2 - 0) = 2 + 0.2 = 2.2$$

Now, $$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

By Euler's Method

$$y_2 = y_1 + hf(x_1, y_1) = 2.2 + (0.1) f(0.1, 2.2)$$

$$= 2.2 + (0.1)(2.2 - 0.1) = 2.2 + 0.21 = 2.41$$

$$y_2 = 2.41$$

Example 2. *Apply Euler's Method to solve* $\frac{dy}{dx} = \frac{y - x}{y + x}$ *at* $x = 0.1$, *given that* $y(0) = 1$.

(P.T.U. May 20

Sol. $\frac{dy}{dx} = \frac{y - x}{y + x}$

$0 \rightarrow 0.1$
$y_0 \quad y_1$

Here $f(x, y) = \frac{y - x}{y + x}$, $x_0 = 0$, $y_0 = 1$

take $h = 0.1$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

By Euler's formula

$$y_1 = y_0 + hf(x_0, y_0) = 1 + (0.1) f(0, 1)$$

$$= 1 + (0.1)\frac{(1 - 0)}{1 + 0} = 1 + 0.1$$

$$y_1 = 1.1$$

and then con...

4 TAYLOR SERIES METHOD

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$$

then the solution $y(x)$ is given by Taylor's series

$$y(x) = y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2!}\,y''(x_0) + ...$$

Example 1. *Apply Taylor's series method to find the value of* y (1.1) *and* y (1.2) *correct to three decimal places given that* $\frac{dy}{dx} = xy^{1/3}$, $y_0(1) = y(1) = 1$ *taking the first three terms of Taylor's expansion.*

(P.T.U. June 2003)

Sol.

$$y' = xy^{1/3}$$

$$y'' = \frac{1}{3}xy^{-2/3}\,y' + y^{1/3}$$

$$y''' = \frac{1}{3}[xy^{-2/3}\,y'' + y^{-2/3}\,y' + xy'^2\left(\frac{-2}{3}\right)y^{-5/3}] + \frac{1}{3}y^{-2/3}\,y'$$

at $x = 1, y = 1$

$$y_0' = 1,\quad y_0'' = 2,\quad y_0''' = \frac{10}{9}$$

By Taylor's series

$$y = y_0 + (x - x_0)\,y_0' + \frac{(x - x_0)^2}{2!}\,y_0'' + ...$$

$\therefore$

$$y = 1 + (x - 1) + \frac{(x-1)^2}{2!}(2) + ...$$

$$y_{(1.1)} = 1 + (1.1 - 1) + \frac{(1.1 - 1)^2}{2!}(2) = 1.11$$

$$y_{(1.2)} = 1 + (1.2 - 1) + (1.2 - 1)^2 = 1.24$$

2.5. RUNGE-KUTTA METHOD

Runge Kutta methods are the most accruate methode so for. There methods give the same results as Taylaois series method upto the term containing n^{th} order derivative and are named as the order of the method.

There methods are prefersed to Taylois series methods as the labour of calculating higher order derivatives is seved.

(a) First Order Runge-kutta Method

Cansider the intial value problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

From Euler's method, we have

$$y_1 = y_0 + h f(x_0, y_0) = y_0 + hy_0^1 \quad \text{...(1)}$$

Also from Taylois series method, we have

$$y_1 = y_0 + hy_0^1 + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' \quad \text{...(2)}$$

From (1) & (2) we see that Euleis method gives the same result as Taylois series method when considered upto first order derivative only.

$\therefore$ Euleis method is the Runge-kutta method of first order.

(b) Second Order Runge-Kutta Method

Consider the intial volue problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

Modified Euleis Method satisfies the Taylois series solution upto second order derivative and hence is the Runge-kutta method of second order.

The second order Runge-kutta method is

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

Where

$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf(x_0 + h, y_0 + k_1)$$

Second order Runge-Kutta method is also called Heun's Method.

(c) Third order Runge Kutta Method

Consider the initial value problem

$$\frac{dy}{dx} = f(x_1 y), y(x_0) = y_0$$

The Third order Runge-Kutta method is

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$

As before, Runge-Kutta third order Method gives the same solution as Taylois series method when considered upto third order derivative. This methord is also called Runge's method.

(d) Runge-Kutta Fourth Order Method

This method is used more often and is called Runge-Kutta method only.

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

The fourth order Runge-Kutta formula is given by

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where
$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$$
$$k_3 = hf(x_0 + h/2, y_0 + k_2/2)$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Example 1. *Use Runge-Kutta method to find the value of y when $x = 0.5$ given that $y = 1$ when $x = 0$ and $\frac{dy}{dx} = \frac{y-x}{y+x}$, take $h = 0.25$.* (P.T.U. Dec. 2002)

Sol.

$$\frac{dy}{dx} = \frac{y-x}{y+x}$$

$$f(x, y) = \frac{y-x}{y+x}, x_0 = 0, y_0 = 1, h = 0.25$$

$$x_1 = 0.25$$
$$k_1 = hf(x_0, y_0) = 0.25 f(0, 1) = 0.25$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.25\, f\left(0 + \frac{0.25}{2}, 1 + \frac{0.25}{2}\right)$$

$$= 0.25\, f(0.125, 1.125) = 0.05$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.25\, f(0.125, 1.025) = 1.1956$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.25\, f(0.25, 1.1956) = 0.2065$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.15795$$

$$y_1 = y_0 + k = 1 + 0.15795 = 1.15795$$

$$x_2 = 0.5$$

Again $\quad k_1 = hf(x_1, y_1) = 0.25\, f(0.25, 1.15795) = 0.16122$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.25\, f(0.375, 1.23856) = 0.13379$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.25\, f(0.375, 1.2248) = 0.1328$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.25\, f(0.5, 1.29075) = 0.1104$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.13413$$

$$y_2 = y_1 + k = 1.15795 + 0.13413 = 1.29925$$

Example 2. *Apply Runge Kutta method to solve the initial value problem* $\frac{dy}{}$ ...

Statistical Methods Unit-4

* Curve Fitting :

Fit a Straight Line (By Method of Least Square)

$$y = ax + b \quad \text{—— (1)}$$

Two Normal Equations are :

$\Sigma y = a\Sigma x + nb$ where $n \rightarrow$ Total no. of observations

$\Sigma xy = a\Sigma x^2 + b\Sigma x$

on solving two Normal equations find the Value of a & b.

Then, Put the Values of a & b in eqn (1)

Example : Fit a curve $y = ax + b$ by method of Least Square for the data :

x : 1 2 3 4 5

y : 14 27 40 55 68

Sol^n :

x	y	x^2	xy
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340
$\Sigma x = 15$	$\Sigma y = 204$	$\Sigma x^2 = 55$	$\Sigma xy = 748$

∴ Two Normal eqns are :

$204 = 15a + 5b$

$748 = 55a + 15b$

on Solving

$204 = 15a + 5b \} \times 3$ 55

$748 = 55a + 15b \} \times 1$ 15

$612 = 45a + \cancel{15b}$

$-748 = 55a + \cancel{15b}$

$+136 = +10a$

$\boxed{a = 13.6}$

$204 = 15a + 5b$

$204 = 15(13.6) + 5b$

$204 = 204 + 5b$

$5b = 0$

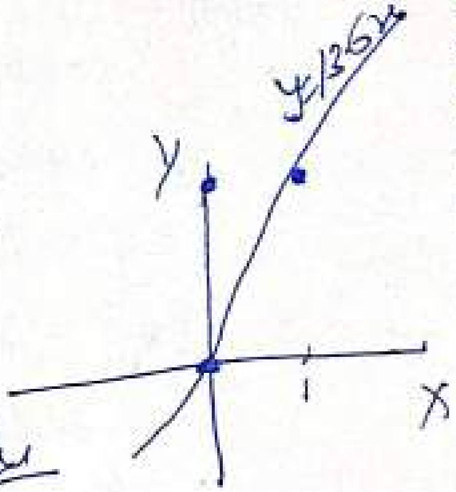
$\boxed{b = 0}$

$\therefore$ $a = 13.6$ & $b = 0$

Put these Values in eqn (1) we get

$y = ax + b$

$y = 13.6x + 0 \Rightarrow \boxed{y = 13.6x}$ Ans



Try

Example-2: Fit a Straight Line to the data:

x:	1	2	3	4	5
y:	1	5	11	8	14

Soln:

x	y	y^2	xy
1	1		
2	5		
3	11		
4	8		
5	14		
$\Sigma x =$	$\Sigma y =$	$\Sigma y^2 =$	$\Sigma xy =$

Ans $a = 2.9$, $b = -0.9$

$\boxed{y = 2.9x - 0.9}$ A

Example 8. *Fit the parabola $y = a + bx + cx^2$ to the data*

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

Sol. Let the Required Curve be

$$y = a + bx + cx^2$$

$\therefore$ normal equations are

$$\Sigma y = c\,\Sigma x^2 + b\,\Sigma x + a \quad ...(1)$$

$$\Sigma xy = c\,\Sigma x^3 + b\,\Sigma x^2 + a\,\Sigma x \quad ...(2)$$

$$\Sigma x^2 y = c\,\Sigma x^4 + b\,\Sigma x^3 + a\,\Sigma x^2 \quad ...(3)$$

x	y	x^2	x^3	x^4	xy	$x^2 y$
0	1	0	0	0	0	0
1	1.8	1	1	1	1.8	1.8
2	1.3	4	8	16	2.6	5.2
3	2.5	9	27	81	7.5	22.5
4	6.3	16	64	256	25.2	100.8
10	**12.9**	**30**	**100**	**354**	**37.1**	**130.3**

equations (1), (2) and (3) becomes

$$12.9 = 30\,c + 10\,b + a \quad ...(4)$$

$$37.1 = 100\,c + 30\,b + 10\,a \quad ...(5)$$

$$130.3 = 354\,c + 100\,b + 30\,a \quad ...(6)$$

On solving (4), (5) and (6)

$$c = 0.55,\ b = 1.07,\ a = 1.42$$

Hence, required curve is

$$y = 0.55\,x^2 + 1.07\,x + 1.42$$

Chapter - 03

-: Correlation (r) :-

Association or Relationship b/w two or more variables.

eg- Income & expenditure.

Types of correlation-

(i) positive corr:- If two variables x & Y move in the same direction than it is called positive correlation.

(ii) Negative corr.:- If two variables x & Y move in the opposite direction.

ex- Demand & supply.

(iii) linear corr.:- If the ratio of change of two variables x & Y remains constant through out.

(iv) Curvilinear/Non-linear corr.:- If ~~time~~ the ratio of change of two variables x & Y remains non-constant through out.

(v) simple corr.:- when ~~which~~ we the study the relationship b/w two variables only that is called simple corr.

(vi) Partial corr.:- when three or more variables are taken but relationship b/w any two variables are ~~dit~~ study. assuming ~~other~~ under variable are constant.
ex. study wheat yield & amount of rainfall under const temp

(vii) Multiple corr.:- when the study the relationship amon

Methods of correlation-
(1) scatter diagram methods
(2) correlation graph
(3) Karl Peason's coeff of correlation.
(4) Rank corr./ spearman's rank corr.
(5) Concurrent deviation method.

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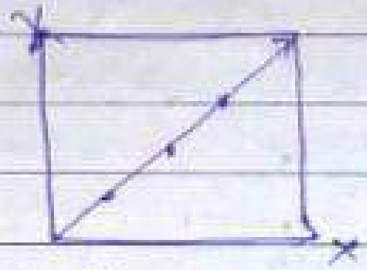
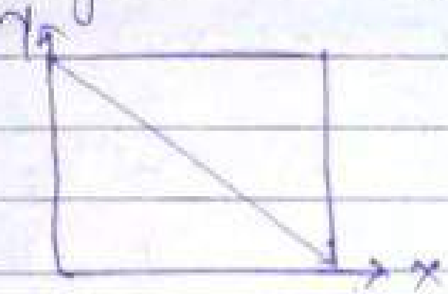
1. scatter diagram method-

(i) perfect positive corr:- If all the points are plotted in the shape of straight line passing from the lower corner of the left side to the upper corner of the right side



(ii) Perfect negative corr.-



(iii) High degree positive corr-

(vi) low degree Negative corr.

(iv) High degree -ve corr.

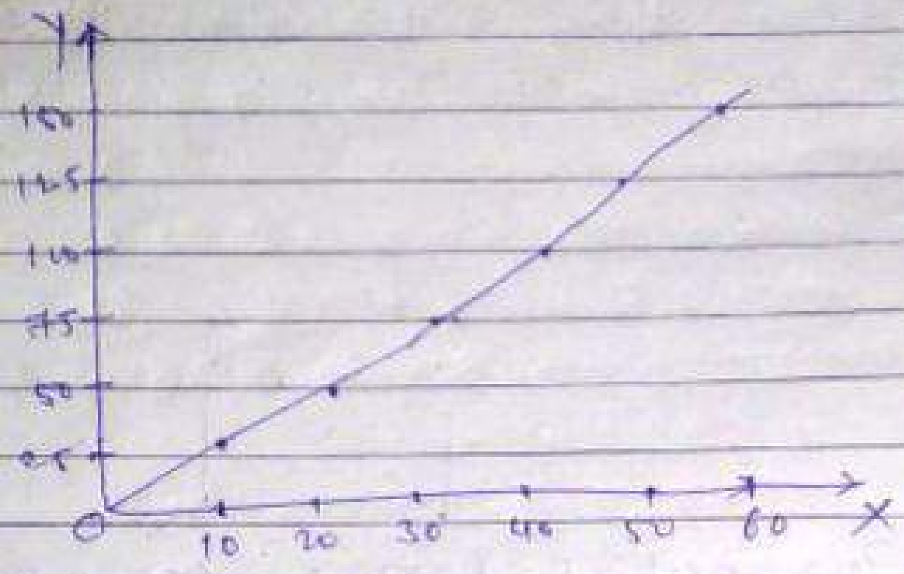
(v) low degree +ve corr.

(vii) zero or No corr.

Ans ① x : 10 20 30 40 50 60

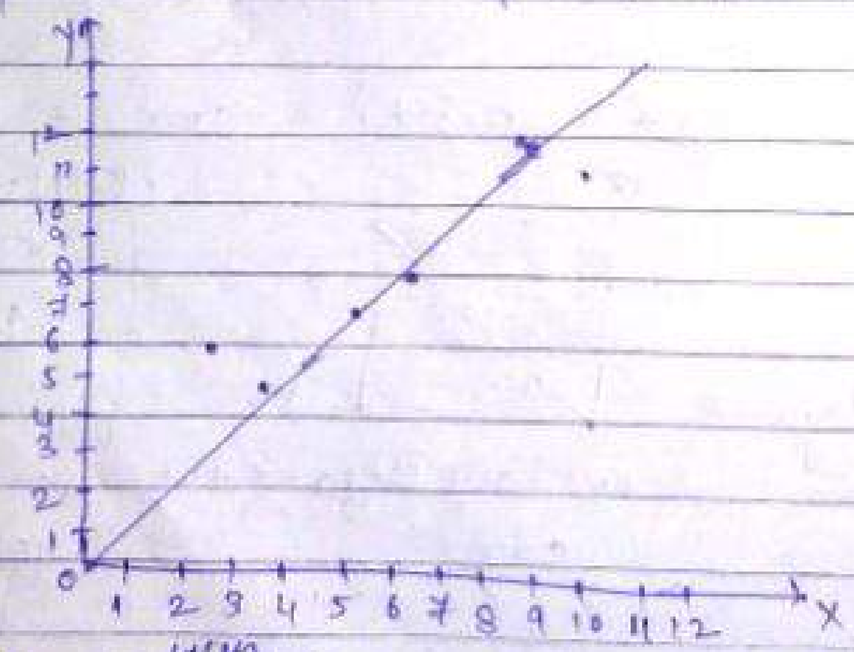
y : 25 50 75 100 125 150

③ make a scatter diagram.
⑤ Is there any corr b/w x & y



It is perfect positive corr.

Ans. ② x : 2 3 5 6 8 9
y : 6 5 7 8 12 11

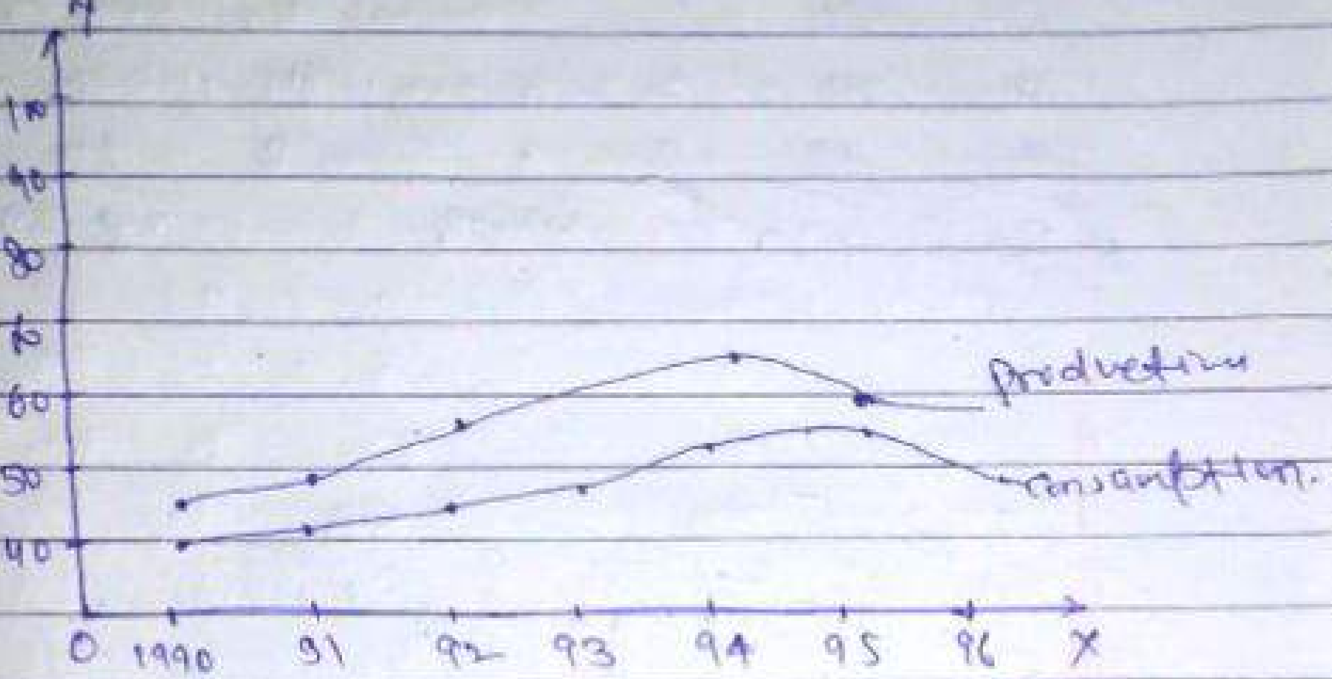


It is high degree positive corr.

(2) Correlation Graph-

Que(1) Construct the correlation graph given data-

Year	1990	1991	1992	1993	1994	1995
Production	46	49	58	58	64	60
Consumption	40	42	54	55	58	57



(3) Karl Pearson's coeff- of Correlation-

(i) Actual mean method-

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \times \Sigma y^2}}$$

where: $x = X - \bar{X}$

$y = Y - \bar{Y}$

Qn ① Find out the karl pearson's of corr. of data-

X	Y	$x = X - \bar{X}$	x^2	$y = Y - \bar{Y}$	y^2	xy
2	4	-3	9	-6	36	18
3	7	-2	4	-3	9	6
4	8	-1	1	-2	4	2
5	9	0	0	-1	1	0
6	10	1	1	0	0	0
7	14	2	4	4	16	8
8	18	3	9	8	64	24
$\bar{X} = \frac{\Sigma X}{N} = \frac{35}{7} = 5$	$\bar{Y} = 10$		$\Sigma x^2 = 28$		$\Sigma y^2 = 130$	$\Sigma xy = 58$

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \times \Sigma y^2}} = \frac{58}{\sqrt{28 \times 130}}$$

$$= \frac{58}{\sqrt{3640}}$$

$$= \frac{58}{60.33}$$

$$= 0.96 \text{ Ans}$$

Q ② Find the coeff. of corr. b/w X & Y –

X	Y	$x = X-\bar{X}$	x^2	$y = Y-\bar{Y}$	y^2	$x \cdot y$
45	35	−16	256	−29	841	464
70	90	9	81	26	676	234
65	70	4	16	6	36	24
30	40	−31	961	−24	576	744
90	95	29	841	31	961	899
40	40	−21	441	−24	576	504
50	60	−11	121	−4	16	44
75	80	14	196	16	256	224
85	80	24	576	16	256	384
60	50	−1	1	−14	196	14
$\bar{X}=61$	$\bar{Y}=64$		$\Sigma x^2 = \cancel{2914}$ 3490		$\Sigma y^2 = \cancel{5927}$ 4390	$\Sigma xy = 3535$

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \times \Sigma y^2}} = \cancel{\frac{3535}{\sqrt{2914 \times 5927}}} = \frac{3535}{\sqrt{3490 \times 4390}}$$

$$= \cancel{\frac{3535}{\sqrt{17271278}}} = \frac{3535}{\sqrt{15321100}}$$

$$= \cancel{\frac{3535}{4155}} = \frac{3535}{3914.21}$$

$$= \cancel{0.85} \text{ Ans} = 0.903 \text{ Ans}$$

20685258

454840

0.77 Ans

$\sqrt{15321100} = 3914.21$

(ii) Assumed mean method —

$$r = \frac{N\sum dxdy - \sum dx \cdot \sum dy}{\sqrt{N\sum dx^2 - (\sum dx)^2}\sqrt{N\sum dy^2 - (\sum dy)^2}}$$

where,

$dx = X - A$, $A \rightarrow$ Assumed value from the given data

$dy = Y - A$

N = Total no. of observations.

X	Y	dx = X − A	dy = Y − A	dx^2	dy^2	dxdy
2	4	−2	−4	4	16	8
3	7	−1	−1	1	1	1
4	8	0	0	0	0	0
5	9	1	1	1	1	1
6	10	2	2	4	4	4
7	14	3	6	9	36	18
8	18	4	10	16	100	40
		$\sum dx = 7$	$\sum dy = 14$	$\sum dx^2 = 35$	$\sum dy^2 = 158$	$= 72$

$$r = \frac{N\sum dxdy - \sum dx \cdot \sum dy}{\sqrt{N\sum dx^2 - (\sum dx)^2} \cdot \sqrt{N\sum dy^2 - (\sum dy)^2}}$$

$$= \frac{7 \times 72 - 7 \times 14}{\sqrt{7 \times 35 - (7)^2} \times \sqrt{7(158) - (14)^2}}$$

$$= \frac{504 - 98}{\sqrt{245 - 49} \times \sqrt{1106 - 196}}$$

$$= \frac{406}{\sqrt{196} \times \sqrt{910}}$$

$$= \frac{406}{14 \times 30.16}$$

$$= \frac{406}{422.24}$$

$$= 0.96 \text{ Ans}$$

(iii) Direct method –

$$r = \frac{N\Sigma XY - \Sigma X \cdot \Sigma Y}{\sqrt{N\Sigma X^2 - (\Sigma X)^2}\sqrt{N\Sigma Y^2 - (\Sigma Y)^2}}$$

X	Y	X^2	Y^2	XY
2	4	4	16	8
3	7	9	49	21
4	8	16	64	32
5	9	25	81	45
6	10	36	100	60
7	14	49	196	98
8	18	64	324	144
$\Sigma X = 35$	$\Sigma Y = 70$	$= 203$	$= 830$	$= 408$

$$r = \frac{7 \times 408 - 35 \times 70}{\sqrt{7 \times 203 - (35)^2} \times \sqrt{7 \times 830 - (70)^2}}$$

$$= \frac{2856 - 2450}{\sqrt{1421 - 1225} \times \sqrt{5810 - 4900}} = \frac{406}{\sqrt{196}\sqrt{910}}$$

$$= \frac{406}{14 \times 30.16}$$

$$= 0.96 \text{ Ans}$$

(4) Spearman's Rank corr. methods-

(i) when ranks are given -

$$\boxed{R = 1 - \frac{6 \Sigma D^2}{N^3 - N}}$$

where, N = Total of observation

D = Diff. b/w rank $(R_1 - R_2)$

Que ① In a fancy dress competition two Judges following rank to the eight participants -

(R_1) judge X	(R_2) judge Y	$D = R_1 - R_2$	D^2
8	7	1	1
7	5	2	4
6	4	2	4
3	1	2	4
2	3	-1	1
1	2	-1	1
5	6	-1	1
4	8	-4	16
			$\Sigma D = 32$

$$R = 1 - \frac{6 \times 32}{512 - 8}$$

$$= 1 - \frac{192}{504}$$

$$= \frac{504 - 192}{504} = 0.619 \text{ Ay}$$

(ii) When ranks are not given —

X	R_1	Y	R_2	$D = R_1 - R_2$	D^2
78	4	84	3	1	1
36	9	51	9	0	0
98	1	91	1	0	0
25	10	60	6	4	~~25~~ 16
75	5	68	4	1	1
82	3	62	5	-2	4
90	2	86	2	0	0
62	7	58	7	0	0
65	6	53	8	-2	4
39	8	47	10	-2	4
					$\Sigma D^2 = 30$

$$R = 1 - \frac{6 \Sigma D^2}{N^3 - N}$$

$$= 1 - \frac{6 \times 30}{1000 - 10}$$

$$= 1 - \frac{180}{990}$$

$$= 0.818 \quad \text{Ans.}$$

(iii) when equal or tied ranks are given-

$$R = 1 - \frac{6\left[\Sigma D^2 + \frac{1}{12}(m^3-m) + \frac{1}{12}(m^3-m) + \dots\right]}{N^3-N}$$

where,

m = No. of repeated rank in the item.

Que ①

x	R_1	Y	R_2	$D=R_1-R_2$	D^2
10	7	16	5.5	1.5	2.25
15	5	18	3.5	1.5	2.25
15	5	16	5.5	-0.5	0.25
20	1	20	1.5	-0.5	0.25
18	2	20	1.5	0.5	0.25
16	3	18	3.5	0.5	0.25
15	5	12	7	-2	4
					$\Sigma D^2 = 9.5$

$\frac{4+5+6}{3} = 5$

$\frac{1+2}{2} = 1.5$

$\frac{3+4}{2} = 3.5$

$\frac{5+6}{2} = 5.5$

$$R = 1 - \frac{6\left[9.5 + \frac{1}{12}(3^3-3) + \frac{1}{12}(2^3-2) + \frac{1}{12}(2^3-2) + \frac{1}{12}(2^3-2)\right]}{343-7}$$

$$= 1 - \frac{6\left[9.5 + 2 + 0.5 + 0.5 + 0.5\right]}{336}$$

$$= 1 - \frac{6 \times 13}{336}$$

$= 0.7678$ Ans

Properties of correlation coeff (r)

(i) $-1 \leq r \leq 1$

(ii) It is symmetric.

$r_{(xy)} = r_{(yx)}$

(iii) $r = \pm\sqrt{b_{xy} \times b_{yx}}$

(iv) Corr. coeff. is independent of change of origin and scale

origin $\pm$

X :	2	3	4	X±2 :	4	5	6
Y :	4	5	6	Y±2 :	6	7	8

$r = \boxed{0.46}$ $r = \boxed{0.46}$

Scale:

X÷ 2X : 4 6 8

2Y : 8 10 12

$r = \boxed{}$

(v) It is independent of unit of measurement.

(vi) It is represent the direction of the relationship.

Chapter - 9

Regression (b) :- Regression is the study of the nature of relationship b/w the variables show that one may be able to predict the unknown value of one variable for a known value of another variable.

Difference b/w corr. and Regression

Corr.	Reg.
(i) Relationship b/w two or more variables	(i) Average relationship b/w two or more variables.
(ii) Its measures Degree.	(ii) Its measures
(iii) It is symmetric. $r_{xy} = r_{yx}$	(iii) It is not symmetric $b_{xy} \neq b_{yx}$
(iv) Independent of origin and Scale	(iv) Independent of origin but not of scale.
(v) Non-sense corr. eg - Income & weight	(v) Nothing like that
(vi) Does not helps in making prediction	(vi) Helps in making prediction.

Method of regression Equation-

(1) Least square method/Normal eqn meth-

Regression eqn of Y on X.

$$Y = a + bX \quad \text{--- (1)}$$

Two Normal eqn are:

$$\Sigma Y = Na + b\Sigma X$$

$$\Sigma XY = a\Sigma X + b\Sigma X^2$$

on solving these eqn. find a & b

put the values of a & b in (1)

Regression eqn of X on Y-

$$X = a + bY \quad \text{--- (1)}$$

Normal eqn are

$$\Sigma X = Na + b\Sigma Y$$

$$\Sigma XY = a\Sigma Y + b\Sigma Y^2$$

Ex (1) Obtain the Reg. eqn for the data-

X	Y	XY	X^2	Y^2
2	12	24	4	144
4	16	64	16	256
6	14	84	36	196
8	12	96	64	144
10	16	160	100	256
$\Sigma X = 30$	$\Sigma Y = 70$	$\Sigma XY = 428$	$\Sigma X^2 = 220$	$\Sigma Y^2 = 996$

Reg. eqn of y on x -

$y = a + bx$ ——— ①

Two normal eqn are.

$70 = 5a + 30b$ ——— ⓐ

$428 = 30a + 220b$ ——— ⓑ

eqn a × 6 -

$420 = 30a + 180b$

$428 = 30a + 220b$

$- \quad - \quad -$

$+8 = +40b$

$b = 0.2$

$70 = 5a + 30 \times 0.2$

$70 = 5a + 6$

$70 - 6 = 5a$

$64 = 5a$

$a = 12.8$

$y = 12.8 + 0.2x$

Reg. eqn of x on y -

$x = a + by$

Two normal eqn are -

$30 = 5a + \cancel{180}\ 70b$ ——— ⓒ

$428 = 70a + 996b$ ——— ⓓ

354

eqⁿ ⓒ × 14 —

$$420 = \cancel{70}a + 980b$$
$$\underline{-\ 428 = \cancel{70}a + 996b}$$
$$-8 = -16b$$
$$b = 0.5$$

puting the eqⁿ c

$$30 = 5a + 70 \times 0.5$$
$$30 = 5a + 35$$
$$30 - 35 = 5a$$
$$-5 = 5a$$
$$a = -1$$

$$X = -1 + 0.5Y \quad \text{Ans}$$

② Reg. eqⁿ using reg. coeff-

(i) Reg. eqⁿ of Y on X -

$$Y - \bar{Y} = byx(X - \bar{X})$$

where, $$\boxed{byx = \frac{N\Sigma XY - \Sigma X\Sigma Y}{N\Sigma X^2 - (\Sigma X)^2}}$$

$$\bar{X} = \frac{\Sigma X}{N}, \quad \bar{Y} = \frac{\Sigma Y}{N}$$

(ii) Reg. eqⁿ of X on Y -

$$X - \bar{X} = bxy(Y - \bar{Y})$$

where, $$\boxed{bxy = \frac{N\Sigma XY - \Sigma Y\Sigma X}{N\Sigma Y^2 - (\Sigma Y)^2}}$$

Q ① before same table -

$$byx = \frac{N\Sigma xy - \Sigma x \Sigma y}{N\Sigma x^2 - (\Sigma x)^2}$$

$$= \frac{5 \times 428 - 30 \times 70}{5 \times 220 - 900}$$

$$= \frac{2140 - 2100}{1100 - 900}$$

$$= \frac{40}{200} = 0.2$$

$$\bar{x} = \Sigma \frac{30}{5} = 6$$

$$\bar{y} = \Sigma \frac{70}{5} = 14$$

$$y - \bar{y} = byx(x - \bar{x})$$
$$y - 14 = 0.2(x - 6)$$
$$y - 14 = 0.2x - 1.2$$
$$\boxed{\begin{array}{l} y - 14 - 1.2 = 0.2 \\ \cancel{y - 15.2 = 0.2} \end{array}}$$
$$y = 0.2x - 1.2 + 14$$
$$y = 0.2x + 12.8$$

$$bxy = \frac{N\Sigma xy - \Sigma y \Sigma x}{N\Sigma y^2 - \Sigma(y)^2}$$

$$= \frac{5 \times 428 - 70 \times 30}{5 \times 996 - 4900}$$

$$= \frac{2140 - 2100}{4980 - 4900}$$

$$= \frac{40}{80}$$

$$= 0.5$$

$x - \bar{x} = b_{xy}(y - \bar{y})$

$x - 6 = 0.5(y - 14)$

$x - 6 = 0.5y - 7$

$x = 0.5y - 7 + 6$

$x = 0.5y - 1$ Ans

Que ① Obtain two reg. eqn using normal eqn method. Also estimate the value of x when y = 10 & value of y when x = 6.

x	y	xy	x^2	y^2
1	2	2	1	4
2	5	10	4	25
3	3	9	9	9
4	8	32	16	64
5	7	35	25	49
$\Sigma x = 15$	$\Sigma y = 25$	$\Sigma xy = 88$	$= 55$	$= 151$

$\Sigma y = Na + b\Sigma x$ —— ①

$\Sigma xy = a\Sigma x + b\Sigma x^2$ —— ⑪

$25 = 5a + 15b$

$88 = 15a + 55b$

$75 = 15a + 45b$

$88 = 15a + 55b$

$-13 = -10b$

$b = 1.3$

$25 = 5a + 15 \times 1.3$

$25 = 5a + 19.5$

$25 - 19.5 = 5a$

$5.5 = 5a$

$a = 1.1$

$y = a + bx$

$y = 1.1 + 1.3x$

$10 = 1.1 + 1.3x$

~~$10 = 3.2x$~~

~~$x =$~~

$y = 1.1 + 1.3 \times 6$

$y = 1.1 + 7.8$

$y = 8.9$

$10 - 1.1 = 1.3x$

$8.9 = 1.3x$

$x = 6.84$

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Que ② # To obtain the reg eqn coeff of corr. (r), S.D (σ) and Airthmatic mean (AM) of x & y are given.

(i) Reg. eqn of Y on X -

$Y - \bar{Y} = b_{yx}(X - \bar{X})$

where,

$$\boxed{b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}}$$

(ii) Reg. eqn of X on Y.

$X - \bar{X} = b_{xy}(Y - \bar{Y})$

where,

$$\boxed{b_{xy} = r \cdot \frac{\sigma_x}{\sigma_y}}$$

Que ① find the reg. eqn of Y on X for data.

	X	Y
A.M	5	12
S.D	2.6	3.6
corr. coeff = 0.7		

Soln. $\bar{x} = 5$, $\bar{y} = 12$

S.D (σ_x) = 2.6, $\sigma_y = 3.6$

$r = 0.7$

Reg eqⁿ of Y in X -

$$b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}$$

$$= 0.7 \times \frac{3.6}{2.6}$$

$$= \frac{2.52}{2.6}$$

$$= 0.96$$

$$Y - \bar{Y} = b_{yx}(X - \bar{X})$$
$$Y - 12 = 0.96(X - 5)$$
$$Y - 12 = 0.96X - 4.8$$
$$Y = 0.96X - 4.8 + 12$$
$$Y = 0.96X + 7.2 \text{ Ans}$$

(i) Estimate the value of Y when $X = 9$

$$Y = 0.96 \times 9 + 7.2$$
$$= 15.84$$

Reg. eqⁿ of X on Y -

$$b_{xy} = r \cdot \frac{\sigma_x}{\sigma_y}$$

$$= 0.7 \times \frac{2.6}{3.6}$$

$$= 0.5$$

$$X - \bar{X} = b_{xy}(Y - \bar{Y})$$
$$X - 5 = 0.5(Y - 12)$$
$$X - 5 = 0.5Y - 6$$
$$X = 0.5Y - 6 + 5$$
$$X = 0.5Y - 1 \text{ Ans}$$

$\boxed{r^2 \le 1}$

To find coeff of corr. ~~was~~ using two regression eqns -

$b_{yx} = +ve \quad , b_{xy} = +ve$

$r^2 = b_{yx} \cdot b_{xy}$

$r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

$r \rightarrow +ve$ | $b_{yx} \rightarrow$ coeff of x

$r \rightarrow -ve$ | $b_{xy} \rightarrow$ coeff of y

Ans ① From two reg. eqns find which one is y on x & which is x on y?

$2x + 3y = 42$ — ①

$x + 2y = 26$ — ②

Soln Let ① eqn be y on x.

$3y = 42 - 2x$

$y = \frac{42}{3} - \frac{2}{3}x$

$b_{yx} = -\frac{2}{3}$

② → x on y

$x = 26 - 2y$

$b_{xy} = -2$

$r^2 = b_{yx} \cdot b_{xy}$

$= -\frac{2}{3} \times -2 = \frac{4}{3} > 1$

our supposition is wrong.

Let assumed ① eqn be x on y.

$$2x = 42 - 3y$$

$$x = \frac{42}{2} - \frac{3}{2}y$$

$$b_{xy} = -\frac{3}{2}$$

② eqn be y on x.

~~$x = 26 - 2y$~~

~~$b_{yx} = -2$~~

~~$r^2 = -\frac{3}{2} \times -2$~~

~~$r = \frac{6}{2}$~~

$$2y = 26 - x$$

$$y = \frac{26}{2} - \frac{1}{2}x$$

$$b_{yx} = -\frac{1}{2}$$

$$r^2 = -\frac{3}{2} \times -\frac{1}{2}$$

$$= \frac{3}{4} \quad \cancel{= 0.75}$$

$$r = -0.86$$

part - ②

Find mean of x and mean of y

$$2x + 3y = 42$$
$$2x + 4y = 52$$
$$- \quad - \quad -$$

$$-y = -10$$

$$\bar{y} = 10$$

$$2x + 3 \times 10 = 42$$

$$2x = 42 - 30$$

$$2x = 12$$

$$\bar{x} = 6$$

Part (3) find the reg coeff

$b_{xy} = -\frac{3}{2}$, $b_{yx} = -\frac{1}{8}$

Part. (2) $4x - 5y + 30 = 0$ —— (1)

$20x - 9y - 107 = 0$ —— (2)

Sol^n let,

eq^n (1) be Y on X.

$-5y = -30 - 4x$

$-y = +\frac{30}{5} + \frac{4}{5}x$

$b_{yx} = +\frac{4}{5}$

(2) eq^n be x on y.

$20x = 107 + 9y$

$x = \frac{107}{20} + \frac{9y}{20}$

$b_{xy} = \frac{9}{20}$

$r^2 = b_{yx} \cdot b_{xy}$

$= \frac{4}{5} \times \frac{9}{20} = \frac{36}{100} = 0.36$, $r = 0.6$

Our supposition is wrong.

let assumed eq^n (1) be x on y

$4x = -30 + 5y$

$= -\frac{30}{4} + \frac{5}{4}y$

$b_{xy} = \frac{5}{4}$

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eqⁿ ② be y on x

$\cancel{-9y = 107 - 20x}$

$\cancel{y = -\frac{107}{9} + \frac{20}{9}x}$

$\cancel{byx = \frac{20}{9}}$

$\cancel{r^2 = \frac{5}{4} \times \frac{20}{9} = \frac{100}{36} = 2.7}$

$\cancel{r = 1.6}$

variance of $x = 3$

$\sigma_x^2 = 3$

$\sigma_x = 1.73$

$byx = r \cdot \frac{\sigma_y}{\sigma_x}$

$\frac{4}{5} = 0.6 \cdot \frac{\sigma_y}{1.73}$

$\boxed{\cancel{\frac{4}{5} \times 1.73 = 0.6\sigma_y}}$

$3\sigma_y = 6.92$

$\sigma_y = \frac{6.92}{3}$

$\sigma_y = 2.3$

$\sigma^2 y = 5.29$

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COLLOCATION METHOD

This method gives alternative techniques for solving a boundary value problem that avoid to find the functional. This method is some times called a residual method. Let us illustrate the concept with the help of a very simple example as below :

Consider this second order linear boundary value problem over $[a, b]$

$$\left.\begin{aligned} y'' + Q(x)\, y &= f(x), \\ y(a) = y_0,\ y(b) &= y_n \end{aligned}\right\} \quad \ldots(1)$$

We begin by defining the residual, R (x) as equal to the left hand side of equation (1) minus the right hand side.

$$R(x) = y'' + Qy - f \quad \ldots(2)$$

We approximate $y(x)$ again with $u(x)$ equal to a sum of trial functions, usually chosen as linearly independent polynomials. We substitute $u(x)$ into R (x) and attempt to make R $(x) = 0$ by a suitable choice of the coefficients in $u(x)$. Normally we cannot do this everywhere in the interval $[a, b]$, so we select several points at which we make R $(x) = 0$. It should be noted that the number of points where we do this must equal the number of unknown coefficients in $u(x)$.

Example 4. *Solve the equation $y'' + y = 3x^2$ with boundary points (0, 0) and (2, 3.5) using collocation method.*

Sol. Given

$$y'' + y = 3x^2 \quad ...(1)$$

and (0, 0), (2, 3.5) are points.

That is $y(0) = 0$, $y(2) = 3.5$

Here $Q = 1$ and $F = 3x^2$

Use polynomial trial functions up to degree 3. If we define $u(x)$ as

$$u(x) = \frac{7x}{4} + C_2(x)(x-2) + C_3(x^2)(x-2) \quad ...(2)$$

The residual is, after substituting $u(x)$ for $y(x)$

$$R(x) = u'' + u - 3x^2 \quad ...(3)$$

which becomes, when we differentiate u twice to get u''

$$u'(x) = \frac{7}{4} + C_2[(x-2) + x] + C_3[x^2 + 2x(x-2)]$$

$$\Rightarrow \quad u'(x) = \frac{7}{4} + C_2(2x-2) + C_3\ (3x^2 - 4x)]$$

$$u''(x) = 2C_2 + C_3(6x-4)$$

$$\therefore \quad R(x) = 2C_2 + C_3(6x-4) + \frac{7x}{4} + C_2(x^2 - 2x) + C_3(x^3 - 2x^2) - 3x^2 \quad ...(4)$$

Since there are two unknown constants, we can make $R(x)$ to be zero at two points in [0, 2]. We do not know which two points will be the best choices, we arbitrarily take as $x = 0.7$ and $x = 1.3$. These points are more or less equally spaced in the interval. Make $R(x) = 0$ for these choices gives a pair of equations in the C's :

From $x = 0.7$:

$$\frac{1090C_2 - 437C_3 - 245}{1000} = 0 \quad ...(a)$$

from $x = 1.3$:

$$\frac{1090C_2 - 2617C_3 - 2795}{1000} = 0 \quad ...(b)$$

Solving equations (*a*) and (*b*) for the C's, we get for $u(x)$.

$$u(x) = \left(\frac{425}{509}\right)x^3 - \left(\frac{61607}{55481}\right)x^2 + \left(\frac{140023}{221924}\right)x \quad ...(5)$$

ollowing table compares the approximation with the exact solution.

x	$y(x)$	$u(x)$	Error
0.00	0.00	0.00	0.00
0.10	0.00	0.05	– 0.05
0.20	0.00	0.09	– 0.09
0.30	0.00	0.11	– 0.11
0.40	0.01	0.13	– 0.12
0.50	0.02	0.14	– 0.13
0.60	0.03	0.16	– 0.13
0.70	0.06	0.18	– 0.12
0.80	0.10	0.22	– 0.12
0.90	0.16	0.28	– 0.12
1.00	0.24	0.36	– 0.11
1.10	0.35	0.32	0.03
1.20	0.49	0.60	– 0.11
1.30	0.67	0.78	– 0.10
1.40	0.90	1.00	– 0.10
1.50	0.17	1.27	– 0.09
1.60	0.50	1.59	– 0.08
1.70	0.90	1.97	– 0.07
1.80	2.36	2.41	– 0.05
1.90	2.89	2.92	– 0.03
2.00	3.50	3.50	– 0.10

We could improve the approximation by using more terms in $u(x)$.