# BTME-602 HEAT TRANSFER 

## 1. Introduction:

Concept of heat transfer, Difference between the subject of "Heat Transfer" and its parent subject "Thermodynamics". Different modes of heat transfer - conditions, convection, and radiation.

## 2. Conduction:

Fouier's law of heat conduction, coefficient of thermal conductivity, effect of temperature and pressure on thermal conductivity of solids, liquids and gases and its measurement. Threedimensional general conduction equation in rectangular, cylindrical and spherical coordinates involving internal heat generation and unsteady state conditions. Derivation of equations for simple one dimensional steady state heat conduction from three dimensional equations for heat conduction though walls, cylinders and spherical shells (simple and composite), electrical analogy of the heat transfer phenomenon in the cases discussed above. Influence of variable thermal conductivity on conduction through simple cases of walls / cylinders and spheres. Equivalent areas, shape factor, conduction through edges and corners of walls and critical thickness of insulation layers on electric wires and pipes carrying hot fluids. Internal generation cases along with some practical cases of heat conduction like heat transfer through piston crown, through under-ground electrical cables/Hot fluid pipes etc and case of nuclear fuel rod with and without cladding. Introduction to unsteady heat transfer, Newtonian heating and cooling of solids; definition and explanation of the term thermal diffusivity. Numerical.

## 3. Theory of Fins:

Concept of fin, classification of fins and their applications. Straight fins of uniform cross-section; e.g. of circular, rectangular or any other cross-section). Straight fins with varying cross-sectional area and having triangular or trapezoidal profile area. Circumferential fins of rectangular crosssection provided on the circumference of a cylinder. Fin performance: fin effectiveness and fin efficiency, total fin effectiveness, total fin efficiency. Optimum design of straight fin of rectangular and triangular profile area. Application of fins in temperature measurement of flow through pipes and determination of error in its measurement. Numerical.

## 4. Convection:

Free and forced convection. Derivation of three-dimensional mass, momentum and energy conservation equations (with introduction to Tensor notations).

Boundary layer formation, laminar and turbulent boundary layers (simple explanation only and no derivation). Theory of dimensional analysis and its application to free and forced convective
heat transfer. Analytical formulae for heat transfer in laminar and turbulent flow over vertical and horizontal tubes and plates. Numerical.

Newton's law of cooling. Overall coefficient of heat transfer. Different design criterion for heat exchangers. Log mean temperature difference for evaporator and condenser tubes, and parallel and counter flow heat exchangers, Calculation of number and length of tubes in a heat exchanger effectiveness and number of transfer units(NTU); Numerical.

## 5. Convection with Phase Change (Boiling and Condensation):

Pool boiling, forced convection boiling, heat transfer during pool boiling of a liquid. Nucleation and different theories of nucleation, different theories accounting for the increased values of h.t.c. during nucleate phase of boiling of liquids; different phases of flow boiling (theory only), Condensation, types of condensation, film wise condensation on a vertical and inclined surface, Numerical.

## 6. Radiation:

Process of heat flow due to radiation, definition of emissivity, absorptivity, reflectivity and transmissivity. Concept of black and grey bodies, Plank's law of nonchromatic radiation. Kirchoff's law and Stefan Boltzman's law. Interchange factor. Lambert's Cosine law and the geometric factor. Intensity of Radiation (Definition only), radiation density, irradiation, radiosity and radiation shields. Derivation formula for radiation exchange between two bodies using the definition of radiosity and irradiation and its application to cases of radiation exchange between three or four bodies (e.g. boiler or other furnaces), simplification of the formula for its application to simple bodies like two parallel surfaces, concentric cylinders and a body enveloped by another body etc. Error in Temperature measurement by a thermocouple probe due to radiation losses.

## Books:

1. Frank P. Incropera and David P. De Witt, Fundamentals of Heat and Mass transfer, John Wiley
2. P.S. Ghoshdastidar, Heat Transfer, Oxford Press
3. D.S. Kumar, Fundamentals of Heat and Mass Transfer, SK Kataria \& Sons ( $6^{\text {th }} / 7^{\text {th }}$ Edition)
4. A.J. Chapman, Heat Transfer, McGraw Hill Book Company, New York.
5. J.P. Holman, Heat Transfer, Tata McGraw-Hill Publishing Company Ltd.(Special Indian Edition).
6. Yunus A.Cengel, Heat and Mass Transfer, Tata McGraw Hills Education Private Ltd (Special Indian Edition).
7. Eckert \& Drake, Heat and Mass Transfer, McGraw Hill Book Company, New York.

## BTME 605 HEAT TRANSFER LAB.

A. Two to three students in a group are required to do one or two practicals in the form of Lab. Project in the topic/s related to the subject matter and in consultation with teacher. The complete theoretical and experimental analysis of the concerned topic is required to be performed (including design and fabrication of new experimental set up, if required, or modifications/retrofitting in the existing experimental set ups). The following topics can be taken as reference:-

1. Determination of thermal conductivity of:

- a solid insulating material by slab method
- powder materials by concentric spheres method / or by some transient heat transfer technique
- a metal by comparison with another metal by employing two bars when kept in series and / or in parallel under different boundary conditions
- Liquids by employing thin layer

2. Determination of coefficient of heat transfer for free/forced convection from the surface of a cylinder / plate when kept:
a) along the direction of flow
b) perpendicular to the direction of flow
c) inclined at an angle to the direction of flow
3. To plot the pool boiling curves for water and to determine its critical point
4. Determination of heat transfer coefficient for
i) film condensation
ii) drop-wise condensation
5. Determination heat transfer coefficient by radiation and hence find the Stefan Boltzman's constant using two plates/two cylinders of same size by making one of the plates/cylinders as a black body.
6. Determination of shape factor of a complex body by an analog technique.
7. To plot the temperature profile and to determine fin effectiveness and fin efficiency for
i) A rod fin when its tip surface is superimposed by different boundary condition like.
a) Insulated tip
b) Cooled tip
c) Temperature controlled tip
ii) Straight triangular fins of various sizes and optimization of fin proportions
iii) Circumferential fins of rectangular/triangular section
B. Each student is required to use Finite Difference Method for analysis of steady state one dimensional and two dimensional conduction problems (Minimum two problems one may be from the Lab. Project) such as conduction through plane/cylindrical/spherical wall with or without internal heat generation, heat transfer through fins, bodies with irregular boundaries subjected to different boundary conditions.

## BTME 606 FLUID MACHINERY LAB

Internal Marks: 30
External Marks: 20
Total Marks: 50

1. Determination of various efficiencies of Hydraulic Ram
2. To draw characteristics of Francis turbine/Kaplan Turbine
3. To study the constructional features of reciprocating pump and to perform test on it for determination of pump performance
4. To draw the characteristics of Pelton Turbine
5. To draw the various characteristics of Centrifugal pump
6. Determine the effect of vane shape and vane angle on the performance of centrifugal fan/Blower
7. A visit to any Hydroelectric Power Station

## Heat Transfer

Heat transfer, also referred to simply as heat, is the movement of thermal energy from one thing to another thing of different temperature.

## Difference between Heat Transfer and Thermodynamics

"Thermodynamics" deals with the amount of energy in form of heat or work during a process and only considers the end states in equilibrium. "Heat Transfer" deals with the rate of energy transfer, thus, it gives idea of how long a heat transfer will occur? Heat transfer deals with time and non equilibrium phenomena.

Thermodynamics gives no indication about how long the process takes. Heat Transfer determines how fast heat can be transferred to or from a system and thus the times of cooling or heating.

# Heat Transfer: Conduction, Convection, and Radiation 

## Introduction

We have learned that heat is the energy that makes molecules move. Molecules with more heat energy move faster, and molecules with less heat energy move slower. We also learned that as molecules heat up and move faster, they spread apart and objects expand (get bigger). This is called thermal expansion.


Heat is always moving! If you have two objects or substances that are different temperatures, heat will always move OUT of the warmer object or substance, and INTO the cooler object or substance. This heat transfer will continue until the objects are the same temperature.


So how, exactly, does heat move out of one thing and into another thing? This is called heat transfer. (Remember, we learned that energy transfer is when energy moves from one thing or place to another, but the energy type stays the same). Heat can transfer (or move) in 3 ways: conduction, convection, and radiation. As you read about the three types of heat transfer, pay attention to:

- What the heat is moving through (solids, liquids and gases, or empty space)
- How the heat is being transferred (touch, currents, or waves)


## Conduction

Last weekend, I went to the beach. I was walking barefoot on the soft, cool grass. When I got to the sand, I noticed that my feet were burning! Ouch! This is an example of conduction.


Conduction is how heat transfers through direct contact with objects that are touching. Any time that two objects or substances touch, the hotter object passes heat to the cooler object. (That hot sand passed the heat energy right into my poor feet!)


Think of a row of dominoes that are all lined up. When you push the first domino, it bumps into the second one, which bumps into the third one...all the way down the line. Heat conduction is like the dominoes. Imagine that you place one end of a metal pole into a fire. The molecules on the fire end will get hot. Each hot molecule will pass the heat along to the molecule next to it, which will pass the heat along to the next molecule, and so on. Before you know it, the heat has traveled all the way along the metal pole until it reaches your hand.


Some materials are better conductors than others. That's because some materials are able to pass (conduct) heat more easily. Metals are great conductors. That's why metal objects get hot so easily. Plastic and wood are poor conductors. They will still get hot, but it takes a lot longer for them to pass the heat from molecule to molecule.

Likewise, solids are better conductors than liquids or gases. That's because solids have molecules that are very tightly packed together, so it's much easier for the molecules to pass the heat along. The molecules in liquids and gases are spread further apart, so they aren't touching as much. It takes longer for liquids and gases to conduct heat.

There are many examples of heat conduction. Any time two object touch, heat conduction will happen. Touching a hot iron is an example of conduction - the heat passes out of the iron and into your hand. So is holding an ice cube - the heat is conducted out of your hand, and into the ice cube (that's why your hand feels cold). Cooking food on the stove is an example of conduction happening twice - the heat from the burner passes into the metal pan, and then the heat from the metal pan passes into the food, heating it up.


## Convection

Convection is how heat passes through fluids. A fluid is anything that has loosely moving molecules that can move easily from one place to another. Liquids and gases are fluids.

One important property of fluids is that they rise when heated. That's because the molecules spread out and move apart when they get hot. The hot fluid becomes less dense and rises up. Cooler fluid is less dense and so it sinks down. This up-and-down motion creates what are called convection currents. Convection currents are circular movements of heated fluids that help spread the heat.


Here's an example. Last night I heated up soup for dinner. Yum! At first, the soup was cold in the pan. The soup at the bottom of the pan was closest to the hot stove burner, right? So the soup at the bottom heated up first. As it heated, the molecules spread apart and became less dense. So the heated soup rose up to the top.

As the hot soup rose up, the cooler soup at the top sank down to the bottom. When it was at the bottom, it was closest to the heat, so THAT soup got hot and rose up. As the soup continued heating, the hot soup rose and the cold soup sank. If you were to look closely, you would see the soup moving up and down in the pot. The up-and-down movement was a convection current. The convection current helped spread the heat around, until all of the soup was heated up.


Convection currents explain why the air is hotter at the top of a room and cooler at the bottom. Convection currents also explain why water is warm at the top of the ocean, but gets colder as you swim deeper.

One natural example of convection currents is wind. As the Sun shines down on an area of land, it heats the air above the ground. That warm air rises. As it rises, cooler air moves in to take the place at the bottom. This moving cooler air creates...wind! Wind happens all over Earth because Earth heats unevenly. There are always colder parts and warmer parts. The wind blows from the cooler parts of Earth to the warmer parts.


Other examples of convection are: boiling a pot of water on the stove; using a hot radiator to warm the air in a room; and using heated air to make a hot-air balloon rise up into the sky.

## Radiation

So we've learned that conduction moves heat easiest through solids, and convection moves heat through liquids and gases. So how does the heat from the Sun get to Earth? There are no molecules in space! And how do you feel the heat from a campfire, even if you're sitting several feet away?

The answer is radiation. Radiation is how heat moves through places where there are no molecules. Radiation is actually a form of electromagnetic energy. Remember we learned that electromagnetic energy moves in waves? Well, radiation is heat moving in waves. Radiation does NOT need molecules to pass the energy along.

All objects radiate heat, but some radiate much more heat than others. The biggest source of radiation is the Sun - it sends a HUGE amount of heat to Earth through electromagnetic waves. (Last weekend, at the beach, I could definitely feel the wonderful heat radiation from the Sun. I guess that's why I got a sunburn. Oops! A little too much radiation!)


Light bulbs radiate heat. Try it! Hold your hand a few inches away from a light bulb. You can feel the heat, right? In fact, a good way to remember radiation is that it is how you can feel heat without touching it. Heat passes through the empty space until it reaches your hand. That's radiation! A fire is another example of radiation. Even YOU are an example. Your body gives off heat! (That's why a classroom gets warm when there are a lot of people sitting in it.)


## Review

Remember - heat always passes from a warmer object to a cooler object until all objects are the same temperature.

Conduction is how heat travels between objects that are touching. Conduction travels fastest through solids, but liquids and gases can also conduct heat. Some materials, like metal, can conduct heat very quickly, while other materials (like plastic or wood) conduct heat very slowly.

Convection is how heat travels through fluids - liquids and gases. Hot fluids rise up, while cold fluids sink down. This up-and-down motion is called a convection current. Convection current spreads the heat in a circular, up-and-down pattern.

Radiation is how heat travels through empty space. Radiation does NOT require molecules to travel through. Any time you can feel heat without touching it, you are experiencing radiation.


$$
\begin{aligned}
\theta_{z} & =1.25 \times(5 \times 0.25) \times \frac{50}{6} \\
& =13.02 \mathrm{w} .
\end{aligned}
$$

Thermal conductivity
(i) It is the ability of the material to conduct heat through it.
(ii) It is the property of material $\alpha$ it's standard value is available in databooks.
(iii) Greater the value of $k$, more will be the heat conduction and the material having higher value of $k$ is frown as conductors whereas the materials having lover value of $k$ is known as insulators.
(iv) Ingenveral, it is highest for the pure metal followed by metallic alloys than non metals ce.

$$
K_{\text {pure }}^{\text {metal }}>K_{\text {Alloys }}>K_{\text {Non-metal }}
$$

(v) In general, it is highest for the solids followed by liquid $\Rightarrow$ gases. ie.

$$
K \text { solid }>K \text { liquid }>K \text { gases }
$$

$$
\underbrace{K_{\text {diamond }}>K_{\text {pure }}^{\text {metal }}}_{\text {solids. }}>K_{\text {alloy }}>K_{\text {Non }}^{\text {metal }} \text { }>K_{\text {liquid }}>K_{\text {gas }}
$$

(Vi) For conductors, the thermal conductivity decreases with increase in temperature whereas for insulators, the thermal conductivity increases with increase in temperature

In conductors, the $k$ is due to -
(a) Lattice vibrations
(b) Movement of free electrons

As leaps
In conductors, there ace large no. of electrons $x$ They hate also systematic arranged molecules but as temp increases the lotic vibration also increases Which creates hindrance in the flow of grue electrons. Hence thermal conductivity decreases.

In the insulators, there is less no. of fee electrons. So due to less no. of fore electrons, $K$ is less.
(Vii) In liquid, the $k$ decreases with increase in temp. In case of gases, $K$ is given by kinetic theory of gases and acc. to that thermal cond. $K$ is directly proportional to $\sqrt{T}$
ie. $K \propto \sqrt{T}$ (For gases)
(and when molecular weight is constant
and $K \propto \frac{1}{\sqrt{M}}$ (when temp is constant)
M-Molewlar weight of gas.
(Viii) The ratio of thermal conductivity to electrical conductivity is constant at a given temp. ie.

$$
\begin{aligned}
& \frac{K}{\sigma} \propto T \\
& \frac{K}{\sigma T}=\text { constant }
\end{aligned}
$$

$\sigma$-electrical cone.
IT-Assolute temp. (inseloin)

This constant $C$ is moon as Lorentz Number.

$$
c=2.45 \times 10^{-8}
$$

(ix) The value of thermal conductivity for sone common materials at room temperature ie. $25^{\circ} \mathrm{C}$
a) Silver: $407 \mathrm{w} / \mathrm{mk}$
b) Copper : $386 \mathrm{wi} m \mathrm{~m}$
c) Aluminium ? $205 \mathrm{\omega} / \mathrm{mk}$
d) Brass: $110 \mathrm{w} / \mathrm{mk}$
e) Cast Iron: $55 \omega / m k$
f) Mild Steel: $38 \mathrm{\omega} / \mathrm{mK}$
g) Stainless steel: $18 \mathrm{w} / \mathrm{mk}$
h) wood: $0.15 \mathrm{w} / \mathrm{mK}_{\mathrm{K}}$
i) Asbestos: $0.095 \mathrm{w} / \mathrm{mk}$
j) Glassurod: $0.03 \mathrm{w} / \mathrm{mk}$

K $\mathrm{Ha}_{\text {ate }}>$ Eater
k) water: 0.51 wink vapours.
l) Air - : $0.022 \mathrm{w} / \mathrm{mk}$

1-D, Steady state Conduction through Plane wall:-

Fowier's law of heat conduction

$$
\begin{gathered}
Q=-K A \frac{d T}{d x} \\
\int_{0}^{L} Q \cdot d x=\int_{T_{1}}^{T_{2}}-K \cdot A \cdot d T \\
Q \cdot L=K \cdot A \cdot\left(T_{1}-T_{2}\right) \\
Q=\frac{K A\left(T_{1}-T_{2}\right)}{L}
\end{gathered}
$$



Heat Transfer
General heat Candention Equation
objective of conduction analyse is iv
(a) to determine the temp. distribution le variation of temp with time $\&$ position. \&
(b) to make computations for heat transfer cts.

Gervelal heat condention eq" in Cartesian cordinates -
Consider the flow of heat through an infinitesimal vdume element oriented in a thee e dimensional co-ordinatesystem. The sides $d x, d y, d z$ have been taken $\|$ to $x, y \& z$-axis.


Let $t=$ temp at the lift face. This temp may be assumed unifpron over the lntri surface.
$\frac{d t}{d x}=$ temp. Changes and sate of change along $x$-direction.
$\left(\frac{\partial t}{\partial x}\right) d x=$ change of temp through distance $d x$.
$t+\left(\frac{\partial t}{\partial x}\right) d x=$ temp. onright side EFGH (at distance dx from beside).
Let $k_{x_{1}} k_{y}, k_{2}=$ Thermal condentivities along $x_{1}$ y \& $z$ direction

Net heat accumulated is der eliventect volume due to conduction of twat from all the corsinale diction $(A)$ + Heat generated withenelemont $(B)=$ Every stored indecelement ( $C$ )

Let $Q=$ Rate of he .t flowimadisetion (W) or ( Energy $\frac{\text { tine }}{Q}$.
$\therefore Q^{\prime}=Q \cdot d z=$ Total teat flow (flex $)$ in that direction (in time $d z$ )
$\binom{\left.\frac{E}{t} \times t\right)=E}{$ (Jones }.
(A) Net heat ceamellated in elemental $V$ dene due toconderetion of heat fromal coordinate directions.
$\therefore$ Quastives of heat flowing in cantal vdunce from left face $A B C D$ during the the interval $d r$ in $X$-directions -

$$
\begin{equation*}
\text { Heat influx }=Q_{x}^{\prime}=-K_{x}(d y+z) \frac{\partial t}{\partial x} \cdot d x \tag{i}
\end{equation*}
$$

During the carve lime intorwel $\partial z$, the the at flowing out of right side of volume ( $E F \mathcal{C}_{1} H$ ) will le -
Heat efflux , $Q_{(x+d x)}^{\prime}==Q_{x}^{\prime}+\frac{\partial}{\partial x}\left(Q_{x}^{\prime}\right) d x$.
$\therefore$ Heat accumulation in element due to the at flaw in $X$-direction.

$$
\begin{aligned}
d Q_{x}^{\prime} & =Q_{x}^{x}-\left[Q_{x}^{\prime}+\frac{\partial}{\partial x}\left(Q_{x}^{\prime}\right) d x\right] \\
& =Q_{x}^{\prime}-Q_{x}^{\prime}-\frac{\partial}{\partial x}\left(Q_{x}^{\prime}\right) d x \\
& =-\frac{\partial}{\partial x}\left(Q_{x}^{\prime}\right) d x \cdot \\
& =-\frac{\partial}{\partial x}\left[-K_{x}(d y d z) \frac{\partial t}{\partial x} \cdot d z\right] d x \\
& =\frac{\partial}{\partial x}\left[K_{x} \frac{\partial t}{\partial x}\right] d x d y d z \cdot d z
\end{aligned}
$$

lIlly in $y$ \& $y$ directions.

$$
\begin{aligned}
& d Q^{\prime} y=\frac{\partial}{\partial y}\left[k_{y} \frac{\partial t}{\partial y}\right] d x \cdot d y \cdot d z \cdot d z \\
& d c_{z}^{\prime}=\frac{\partial}{\partial z}\left[k_{z} \frac{\partial t}{\partial z}\right] d x \cdot d y \cdot d z \cdot d z
\end{aligned}
$$

Net teat accumulates in the element due to conduction (2) of level from all directions.

$$
\begin{align*}
& A=d Q_{x}^{\prime}+d Q_{y}+d Q_{z}^{\prime} \\
\therefore A= & \frac{\partial}{\partial x}\left[K_{x} \frac{\partial t}{\partial x}\right] d x \cdot d y \cdot d z \cdot d z+\frac{\partial}{\partial y}\left[K_{y} \frac{\partial t}{\partial y}\right] d x \cdot d y \cdot d z \cdot d z+ \\
& \frac{\partial}{\partial z}\left[K_{z} \frac{\partial t}{\partial z}\right] d x \cdot d y \cdot d z \cdot d r \\
= & {\left[\frac{\partial}{\partial x}\left(K_{x} \frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{y} \frac{\partial t}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_{2} \frac{\partial t}{\partial z}\right)\right] d x \cdot d y \cdot d z \cdot d r } \tag{I}
\end{align*}
$$

(B) Total teat generated within the element ( $\mathbb{C} / \mathrm{g})$.

$$
\therefore Q_{g}^{\prime}=q_{g}(d x \cdot d y \cdot d z) \cdot d z
$$

$q_{g}=$ Heat generated / per unit time per unit volume. $\left(\frac{\text { Joules }}{t \cdot m^{3}}\right)$

$$
\therefore\left[\begin{array}{rl}
Q_{g}^{\prime} & =q_{g} \times \text { volume } \times \text { time } \\
& =\frac{\text { watt }}{x^{3}} \times m^{3} \times \sec =\frac{\text { Joules }}{5 e 4} \times \sec =\frac{\text { Joules }}{}
\end{array}\right.
$$

$$
\text { or }\left(\frac{\text { watt }}{m^{3}}\right)
$$

(c) Enugg stored in the element

It is given by increase in thermel energy, ( $\Delta U$ )

$$
\begin{aligned}
& \Delta U=m c_{p} d t \\
& \text { ar } \Delta U=\operatorname{Mass} \times c_{p} x d t \\
& =\text { Density } X V \text { hume } X C_{p} X \text { de in temp. } \\
& \Delta u \left\lvert\, C=\rho(d x \cdot d y \cdot d z) \cdot C_{p} \cdot \frac{\partial t}{\partial z} \cdot d z\right. \\
& \therefore \quad A+B=C \\
& \therefore\left[\frac{\partial}{\partial x}\left(k_{x} \frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_{y} \frac{\partial t}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_{z} \frac{\partial t}{\partial z}\right)\right] d x \cdot d y \cdot d z \cdot d z+\left(q_{y}(d x d y d z) d z,\right. \\
& \pm \rho(d x \cdot d y \cdot d z) \cdot C \frac{\partial t}{\partial z} \cdot d z \\
& d x \cdot d y \cdot d z d r\left[\frac{\partial}{\partial x}\left(k_{x} \frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_{y} \frac{\partial t}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_{2} \frac{\partial t}{\partial z}\right)\right]+q_{y}=d x \cdot d y t_{y} d z \\
& \frac{\partial}{\partial x}\left(K_{x} \frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{y} \frac{\partial t}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_{2} \frac{\partial t}{\partial z}\right)+q_{g}=\rho c \frac{\partial t}{\partial z} \\
& \left(\rho C \frac{\partial t}{\partial r}\right. \text { ) }
\end{aligned}
$$

This eq" is frown as general heat conduction eq" for nen-iurnogenous mataial, self feat generation \& unstedy thee dimensional heal flow.
For constant thermal conductivity,

$$
\begin{aligned}
\therefore k_{x}=k_{y}=k_{2}=k . \quad \text { \& eq } \\
\begin{aligned}
\frac{\partial^{2} t}{\partial x^{2}}+\frac{\partial^{2} t}{\partial y^{2}}+\frac{\partial^{2} t}{\partial z^{2}}+\frac{q g}{k} & =\frac{p c}{k} \frac{\partial t}{\partial z} \\
& =\frac{1}{\alpha} \frac{\partial t}{\partial z}
\end{aligned}
\end{aligned}
$$

where $\alpha=\frac{K}{P C}$ is known as thermal diffusivity.

$$
\alpha=\frac{k}{\rho C}=\frac{\text { Thermal condenturly }}{\text { Thermal capacity }}
$$

Different cases. of general heat en
(i) For ease, when no internal heat generation,

$$
\begin{gathered}
q g=0 \\
\frac{\partial^{2} t}{\partial x^{2}}+\frac{\partial^{2} t}{\partial y^{2}}+\frac{\partial^{2 t}}{\partial z^{2}}=\frac{1}{\alpha} \frac{\partial t}{\partial z} \quad \text { (fourie's eq). }
\end{gathered}
$$

(ii) When temp doesn't depend on time means conduction taker plain instealy state. $\left(\therefore \frac{\partial t}{\partial z}=0\right)$.

$$
\therefore \frac{\partial^{2} t}{\partial x^{2}}+\frac{\partial^{2} t}{\partial y^{2}}+\frac{\partial^{2} t}{\partial z^{2}}+\frac{q y}{k}=0 \quad \text { (Poisson's eq'). }
$$

(iii) When conduction takerplace unsteady state \& no internal heat genclation.

$$
\begin{aligned}
& \therefore \frac{\partial t}{\partial \tau}=0, \quad q z=0 \\
& \therefore \frac{\partial^{2} t}{\partial x^{2}}+\frac{\partial^{2 t}}{\partial y^{2}}+\frac{\partial^{2} t}{\partial z^{2}}=0 \quad(\text { Laplace eq") }
\end{aligned}
$$

(iv) ore dimensional \& stall state conduction with no heatgeneration, $\frac{\partial^{2} t}{\partial y^{2}}-\frac{\partial^{2 t}}{\partial z^{2}}=0, q_{g}=0, \frac{\partial t}{\partial z}=0$

$$
\therefore \frac{\partial^{2} t}{\partial x^{2}}=0
$$

where $\sigma=$ stefan boltzman constant
$A=$ surface area of the body
$T=$ Absolute temp. (in kelvin)
One dimensional steady State Conduction: $\rightarrow$
Fourier's law of heat conduction -It states that litre heat
transfer rate in any direction is directly proportional to -
(i) temperature gradient along the direction of heat transfer.
(ii) the area of the surface 1 to direction of heat transfer.

$$
\therefore \quad Q \propto \frac{d T}{d x}
$$

$$
\text { and } \quad Q \propto A
$$

$$
\therefore \quad Q \alpha A \frac{d T}{d x}
$$

$$
Q=-K \cdot A \cdot \frac{d T}{d x}
$$

(Final - initial) is always To remove - ve of $A T$ Put -vesign before $k$.

离
Assumptions in Fourier's law of heat conduction:-
(i) Steady state condition (temp is constant w.r.t. time).
2) One-dinensional heat transfer. (When the heat bansfor in a particular direction is veryvery large as compared to other tiro, thenit is prion as 1-D heat transfer or when one of the dimensions is very very small as compared to other two dimensions, egg. is $50 \mathrm{~m} \times 40 \mathrm{~m} \times 10 \mathrm{~cm}$.

So, 10 cm is very very small., so it is 1D.
(b) $15 \mathrm{~cm} \times 20 \mathrm{~cm} \times 1.5 \mathrm{~mm}$ 1.5 mm is very very small, so it is 10 .
(3) The material is homogeneous $D$ isotropic. (The value of thermal and. $K$ is uniform throughout the material).
(e)es-1.

A wall of $5 \mathrm{~m} \times 6 \mathrm{~m} \times 25 \mathrm{~cm}$ (thickness) has its left bottom and front surface at $80^{\circ} \mathrm{C}$ and right top $\Delta$ back surface is at $30^{\circ} \mathrm{C}$. If $K=1.25 \mathrm{w} / \mathrm{mK}$ Calculate the heat transfer rate in all directions

Sol.


$$
\begin{aligned}
& \quad Z T=\left(80^{\circ}-30^{\circ}\right)=50^{\circ} \mathrm{C} . \\
& Q=K A \frac{\Delta T}{L} \\
& \therefore \theta_{x}=1.25 \times(5 \times 6) \times \frac{50}{0.25}=7500 \mathrm{~W} \\
& \theta_{y}=1.25 \times(0.25 \times 6) \times \frac{50}{5}=18.75 \mathrm{~W} .
\end{aligned}
$$

## Electrical Analogy of Heat Conduction

Heat conduction in solids is similar to the conduction of electricity in electrical conductors in many aspects. In a conductor, the flow of electricity is driven by a potential difference and so is the flow of heat driven by a difference in temperature. In electric conduction, electric charge is transported from one point in a conductor to another by the motion of the electrons, in thermal conduction, heat is transported from one point of a solid to another by the vibration of the molecules of the solid due to their increased energy.

Heat conduction is governed by Fourier's law which states that: the rate of heat transfer (Q) between two sufficiently close points in a medium is proportional to the temperature difference between the two points ( $\mathrm{T}_{1}-\mathrm{T}_{2}$ ) divided by their separation ( $\Delta \mathrm{x}$ ) and the area normal to the direction of heat flow (A). The constant of proportionality is called the coefficient of thermal conductivity of the material (k). Mathematically, this statement can be written as:
$\mathrm{Q}=\mathrm{kA}\left(\mathrm{T}_{1}-\mathrm{T}_{2}\right) / \Delta \mathrm{x}$
where Q is the rate of heat transfer in Watts.
Electric current flow is governed by Ohm's law, which states that the electric current (I) flowing between two points in a conductor equals the potential difference between the two points $\left(\mathrm{V}_{1}-\mathrm{V}_{2}\right)$ divided by the electric resistance between them (R). Ohm's law can be written as:
$\mathrm{I}=\left(\mathrm{V}_{1}-\mathrm{V}_{2}\right) / \mathrm{R}$
The above equations suggest that the following analogy holds:

## Thermal Conduction <br> Electric Conduction

Heat flow rate: Q
Electric current : I
Temperature difference: $\Delta \mathrm{T}=\mathrm{T}_{1}-\mathrm{T}_{2} \quad$ Potential difference : $\Delta \mathrm{V}=\mathrm{V}_{1}-\mathrm{V}_{2}$
Thermal resistance: $\mathrm{R}_{\text {thermal }}=\Delta \mathrm{x} /(\mathrm{kA}) \quad$ Electric resistance $=\mathrm{R}$


### 2.3. GENERAL HEAT CONDUCTION EQUATION IN CYLINDRICAL COORDINATES

While dealing with problems of conduction of heat through systems having cylindrical geometries (e.g., rods and pipes) it is convenient to use cylindrical coordinates.

Consider an elemental volume having the coordinates ( $r, \phi, z$ ), for three-dimensional heat conduction analysis, as shown in Fig. 2.2.


Fig. 2.2. Elemental volume for three-dimensional heat conduction analysis - Cylindrical coordinates.
The volume of the element $=r d \phi . d r . d z$
Let, $q_{g}=$ Heat generation (uniform) per unit volume per unit time.
Further, let us assume that $k$ (thermal conductivity), $\boldsymbol{\rho}$ (density), $c$ (specific heat) do not alter with position.
A. Net heat accumulated in the element due to conduction of heat from all the coordinate directions considered:
Heat flow in radial direction ( $x-\phi$ ) plane :
Heat influx,

$$
\begin{equation*}
Q_{r}^{\prime}=-k(r d \phi \cdot d z) \frac{\partial t}{\partial r} \cdot d \tau \tag{i}
\end{equation*}
$$

Heat efflux,

$$
\begin{equation*}
Q_{(r+d r)}^{\prime}=Q_{r}^{\prime}+\frac{\partial}{\partial r}\left(Q_{r}\right) d r \tag{ii}
\end{equation*}
$$

$\therefore$ Heat accumulation in the element due to heat flow in radial direction,

$$
\begin{aligned}
d Q_{r}^{\prime} & =Q_{r}^{\prime}-Q_{(r+d r)}^{\prime} \\
& =-\frac{\partial}{\partial r}\left(Q_{r}^{\prime}\right) d r \\
& =-\frac{\partial}{\partial r}\left[-k(r d \phi \cdot d z) \frac{\partial t}{\partial r} \cdot d \tau\right] d r \\
& =k(d r \cdot d \phi \cdot d z) \frac{\partial}{\partial r}\left(r \cdot \frac{\partial t}{\partial r}\right) d \tau \\
& =k(d r \cdot d \phi \cdot d z)\left(r \frac{\partial^{2} t}{\partial r^{2}}+\frac{\partial t}{\partial r}\right) d \tau
\end{aligned}
$$

$$
\begin{equation*}
=k(d r . r d \phi \cdot d z)\left[\frac{\partial^{2} t}{\partial r^{2}}+\frac{1}{r} \frac{\partial t}{\partial r}\right] d \tau \tag{2.16}
\end{equation*}
$$

Heat flow in tangential direction $(r-z)$ plane :
Heat influx,

$$
\begin{equation*}
Q_{\phi}^{\prime}=-k(d r . d z) \frac{\partial t}{r . \partial \phi} d \tau \tag{iii}
\end{equation*}
$$

Heat efflux,

$$
\begin{equation*}
Q_{(\phi+d \phi)}^{\prime}=Q_{\phi}^{\prime}+\frac{\partial}{r \cdot \partial \phi}\left(Q_{\phi}^{\prime}\right) r d \phi \tag{iv}
\end{equation*}
$$

Heat accumulated in the element due to heat flow in tangential direction,

$$
\begin{aligned}
d Q_{\phi}^{\prime} & =Q_{\phi}^{\prime}-Q_{(\phi+d \phi)}^{\prime} \\
& =-\frac{\partial}{r \cdot \partial \phi}\left(Q_{\phi}\right) r \cdot d \phi \\
& =-\frac{\partial}{r \cdot \partial \phi}\left[-k(d r \cdot d z) \frac{\partial t}{r \cdot \partial \phi} \cdot d \tau\right] r \cdot d \phi \\
& =k(d r \cdot d \phi \cdot d z) \frac{\partial}{\partial \phi}\left(\frac{1}{r} \cdot \frac{\partial t}{\partial \phi}\right) d \tau \\
& =k(d r \cdot r d \phi \cdot d z) \frac{1}{r^{2}} \cdot \frac{\partial^{2} t}{\partial \phi^{2}} \cdot d \tau
\end{aligned}
$$

Heat flow in axial direction ( $r$-ф plane) :
Heat influx, $\quad Q_{z}^{\prime}=-k(r . d \phi . d r) \frac{\partial t}{\partial z} d \tau$
Heat efflux, $Q_{(z+d z)}^{\prime}=Q_{z}^{\prime}+\frac{\partial}{\partial z}\left(Q_{z}^{\prime}\right) d z$
Heat accumulated in the element due to heat flow in axial direction,

$$
\begin{align*}
d Q_{z}^{\prime} & =Q_{z}^{\prime}-Q_{(z+d z)}^{\prime} \\
& =-\frac{\partial}{\partial z}\left[-k(r \cdot d \phi \cdot d r) \frac{\partial t}{\partial z} \cdot d \tau\right] d z \\
& =k(d r \cdot r d \phi \cdot d z) \frac{\partial^{2} t}{\partial z^{2}} \cdot d \tau \tag{2.18}
\end{align*}
$$

Net heat accumulated in the element

$$
\begin{equation*}
=k \cdot d r \cdot r d \phi \cdot d z\left[\frac{\partial^{2} t}{\partial r^{2}}+\frac{1}{r} \cdot \frac{\partial t}{\partial r}+\frac{1}{r^{2}} \cdot \frac{\partial^{2} t}{\partial \phi^{2}}+\frac{\partial^{2} t}{\partial z^{2}}\right] d \tau \tag{2.19}
\end{equation*}
$$

B. Heat generated within the element $\left(Q_{g}^{\prime}\right)$ :

The total heat generated within the element is given by

$$
\begin{equation*}
Q_{g}^{\prime}=q_{g}(d r \cdot r d \phi \cdot d z) \cdot d \tau \tag{2.20}
\end{equation*}
$$

C. Energy stored in the element :

The increase in thermal energy in the element is equal to

$$
\begin{equation*}
=\rho(d r \cdot r d \phi \cdot d z) \cdot c \cdot \frac{\partial t}{\partial \tau} \cdot d \tau \tag{2.21}
\end{equation*}
$$

Now, $\quad(A)+(B)=(C)$
... Energy balance/equation

$$
\begin{aligned}
& \therefore \quad k \cdot d r \cdot r d \phi \cdot d z\left[\frac{\partial^{2} t}{\partial r^{2}}+\frac{1}{r} \cdot \frac{\partial t}{\partial r}+\frac{1}{r^{2}} \cdot \frac{\partial^{2} t}{\partial \phi^{2}}+\frac{\partial^{2} t}{\partial z^{2}}\right] d \tau+q_{g}(d r \cdot r d \phi \cdot d z) \cdot d \tau \\
& =\rho(d r \cdot r d \phi \cdot d z) \cdot c \cdot \frac{\partial t}{\partial \tau} \cdot d \tau
\end{aligned}
$$

## 30 Heat and Mass Transfer

Dividing both sides by $d r . r d \phi . d z . d \tau$, we have

$$
\begin{gather*}
k\left[\frac{\partial^{2} t}{\partial r^{2}}+\frac{1}{r} \cdot \frac{\partial t}{\partial r}+\frac{1}{r^{2}} \cdot \frac{\partial^{2} t}{\partial \phi^{2}}+\frac{\partial^{2} t}{\partial z^{2}}\right]+q_{g}=\rho \cdot c \cdot \frac{\partial t}{\partial \tau} \\
\text { or, } \quad\left[\frac{\partial^{2} t}{\partial r^{2}}+\frac{1}{r} \cdot \frac{\partial t}{\partial r}+\frac{1}{r^{2}} \cdot \frac{\partial^{2} t}{\partial \phi^{2}}+\frac{\partial^{2} t}{\partial z^{2}}\right]+\frac{q_{g}}{k}=\frac{\rho c}{k} \cdot \frac{\partial t}{\partial \tau}=\frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \tag{2.22}
\end{gather*}
$$

Equation (2.22) is the general heat conduction equation in cylindrical coordinates.
In case there are no heat sources present and the heat flow is steady and one-dimensional, then eqn. (2.22) reduces to

$$
\begin{array}{ll}
\frac{\partial^{2} t}{\partial r^{2}}+\frac{1}{r} \cdot \frac{\partial t}{\partial r} & =0  \tag{2.23}\\
\text { or, } & \frac{\partial^{2} t}{\partial r^{2}}+\frac{1}{r} \cdot \frac{d t}{d r}
\end{array}=0
$$

Since $\frac{1}{r} \neq 0$, therefore,

$$
\begin{equation*}
\frac{d}{d r}\left(r \cdot \frac{d t}{d r}\right) \text { or } r \cdot \frac{d t}{d r}=\mathrm{constant} \tag{2.24}
\end{equation*}
$$

Equation (2.22) can also be derived by transformation of coordinates, as follows :

$$
x=r \cos \phi, y=r \sin \phi \text { and } z=z
$$

Now, by chain rule :
or,

$$
\begin{align*}
\frac{\partial t}{\partial r} & =\frac{\partial t}{\partial x} \cdot \frac{\partial x}{\partial r}+\frac{\partial t}{\partial y} \cdot \frac{\partial y}{\partial r}=\frac{\partial t}{\partial x} \cos \phi+\frac{\partial t}{\partial y} \sin \phi \\
\cos \phi \frac{\partial t}{\partial r} & =\cos ^{2} \phi \cdot \frac{\partial t}{\partial x}+\sin \phi \cdot \cos \phi \cdot \frac{\partial t}{\partial y} \tag{i}
\end{align*}
$$

(Multiplying both sides by $\cos \phi$ )
Also,

$$
\frac{\partial t}{\partial \phi}=\frac{\partial t}{\partial x} \cdot \frac{\partial x}{\partial \phi}+\frac{\partial t}{\partial y} \cdot \frac{\partial y}{\partial \phi}=\frac{\partial t}{\partial x}(-r \sin \phi)+\frac{\partial t}{\partial y}(r \cos \phi)
$$

or,

$$
\begin{equation*}
\frac{\sin \phi}{r} \cdot \frac{\partial t}{\partial \phi}=-\sin ^{2} \phi \frac{\partial t}{\partial x}+\sin \phi \cdot \cos \phi \cdot \frac{\partial t}{\partial y} \tag{ii}
\end{equation*}
$$

(Multiplying both sides by $\frac{\sin \phi}{r}$ )
From Eqns. (i) and (ii), we have

$$
\begin{align*}
\frac{\sin \phi}{r} \cdot \frac{\partial t}{\partial \phi} & =-\sin ^{2} \phi \frac{\partial t}{\partial x}+\left[\cos \phi \cdot \frac{\partial t}{\partial r}-\cos ^{2} \phi \frac{\partial t}{\partial x}\right] \\
& =-\frac{\partial t}{\partial x}+\cos \phi \frac{\partial t}{\partial r} \\
\therefore \quad \frac{\partial t}{\partial x} & =\cos \phi \frac{\partial t}{\partial r}-\frac{\sin \phi}{r} \cdot \frac{\partial t}{\partial \phi} \tag{iii}
\end{align*}
$$

Differentiating both sides with respect to $x$, we have

$$
\frac{\partial}{\partial x}\left(\frac{\partial t}{\partial x}\right)=\frac{\partial}{\partial x}\left[\cos \phi \cdot \frac{\partial t}{\partial r}-\frac{\sin \phi}{r} \cdot \frac{\partial t}{\partial \phi}\right]
$$

or, $\quad \frac{\partial^{2} t}{\partial x^{2}}=\cos \phi \cdot \frac{\partial}{\partial r}\left(\frac{\partial t}{\partial x}\right)-\frac{\sin \phi}{r} \cdot \frac{\partial}{\partial \phi}\left(\frac{\partial t}{\partial x}\right)$

$$
=\cos \phi \cdot \frac{\partial}{\partial r}\left(\cos \phi \cdot \frac{\partial t}{\partial r}-\frac{\sin \phi}{r} \cdot \frac{\partial t}{\partial \phi}\right)-\frac{\sin \phi}{r} \cdot \frac{\partial}{\partial \phi}\left(\cos \phi \cdot \frac{\partial t}{\partial r}-\frac{\sin \phi}{r} \cdot \frac{\partial t}{\partial \phi}\right)
$$

[Substituting the value of $\frac{\partial t}{\partial x}$ from (iii)]

$$
\begin{equation*}
=\cos ^{2} \phi \cdot \frac{\partial^{2} t}{\partial r^{2}}-\frac{\cos \phi \cdot \sin \phi}{r^{2}} \cdot \frac{\partial t}{\partial \phi}+\frac{\sin ^{2} \phi}{r} \cdot \frac{\partial t}{\partial r}+\frac{\sin ^{2} \phi}{r^{2}} \cdot \frac{\partial^{2} t}{d \phi^{2}}+\frac{\sin \phi \cdot \cos \phi}{r^{2}} \cdot \frac{\partial t}{\partial \phi} \tag{iv}
\end{equation*}
$$

Similarly, $\frac{\partial^{2} t}{\partial y^{2}}=\sin ^{2} \phi \cdot \frac{\partial^{2} t}{\partial r^{2}}+\frac{\cos ^{2} \phi}{r} \cdot \frac{\partial t}{\partial r}-\frac{\cos \phi \cdot \sin \phi}{r^{2}} \cdot \frac{\partial t}{\partial \phi}+\frac{\cos ^{2} \phi}{r^{2}} \cdot \frac{\partial^{2} t}{\partial \phi^{2}}-\frac{\cos \phi \cdot \sin \phi}{r^{2}} \cdot \frac{\partial t}{\partial \phi}$
By adding (iii) and (iv), we get

$$
\frac{\partial^{2} t}{\partial x^{2}}+\frac{\partial^{2} t}{\partial y^{2}}=\frac{\partial^{2} t}{\partial r^{2}}+\frac{1}{r} \cdot \frac{\partial t}{\partial r}+\frac{1}{r^{2}} \cdot \frac{\partial^{2} t}{\partial \phi^{2}}
$$

Substituting it in eqn (2.8), we get,

$$
\left[\frac{\partial^{2} t}{\partial r^{2}}+\frac{1}{r} \cdot \frac{\partial t}{\partial r}+\frac{1}{r^{2}} \cdot \frac{\partial^{2} t}{\partial \phi^{2}}+\frac{\partial^{2} t}{\partial z^{2}}\right]+\frac{q_{g}}{k}=\frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau}
$$

which is the same as eqn. (2.22)

### 2.4. GENERAL HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATES

Consider an elemental volume having the coordinates $(r, \phi, \theta)$, for three dimensional heat conduction analysis, as shown in Fig. 2.3.


Fig. 2.3. Elemental volume for three-dimensional heat conduction analysis - Spherical coordinates.
The volume of the element $=d r \cdot r d \theta \cdot r \sin \theta d \phi$
Let,

$$
q_{g}=\text { Heat generation (uniform) per unit volume per unit time. }
$$

Further let us assume that $k$ (thermal conductivity), $\rho$ (density), $c$ (specific heat) do not alter with position.
or, $\quad \frac{\partial^{2} t}{\partial x^{2}}=\cos \phi \cdot \frac{\partial}{\partial r}\left(\frac{\partial t}{\partial x}\right)-\frac{\sin \phi}{r} \cdot \frac{\partial}{\partial \phi}\left(\frac{\partial t}{\partial x}\right)$

$$
=\cos \phi \cdot \frac{\partial}{\partial r}\left(\cos \phi \cdot \frac{\partial t}{\partial r}-\frac{\sin \phi}{r} \cdot \frac{\partial t}{\partial \phi}\right)-\frac{\sin \phi}{r} \cdot \frac{\partial}{\partial \phi}\left(\cos \phi \cdot \frac{\partial t}{\partial r}-\frac{\sin \phi}{r} \cdot \frac{\partial t}{\partial \phi}\right)
$$

[Substituting the value of $\frac{\partial t}{\partial x}$ from (iii)]

$$
\begin{equation*}
=\cos ^{2} \phi \cdot \frac{\partial^{2} t}{\partial r^{2}}-\frac{\cos \phi \cdot \sin \phi}{r^{2}} \cdot \frac{\partial t}{\partial \phi}+\frac{\sin ^{2} \phi}{r} \cdot \frac{\partial t}{\partial r}+\frac{\sin ^{2} \phi}{r^{2}} \cdot \frac{\partial^{2} t}{d \phi^{2}}+\frac{\sin \phi \cdot \cos \phi}{r^{2}} \cdot \frac{\partial t}{\partial \phi} \tag{iv}
\end{equation*}
$$

Similarly, $\frac{\partial^{2} t}{\partial y^{2}}=\sin ^{2} \phi \cdot \frac{\partial^{2} t}{\partial r^{2}}+\frac{\cos ^{2} \phi}{r} \cdot \frac{\partial t}{\partial r}-\frac{\cos \phi \cdot \sin \phi}{r^{2}} \cdot \frac{\partial t}{\partial \phi}+\frac{\cos ^{2} \phi}{r^{2}} \cdot \frac{\partial^{2} t}{\partial \phi^{2}}-\frac{\cos \phi \cdot \sin \phi}{r^{2}} \cdot \frac{\partial t}{\partial \phi}$
By adding (iii) and (iv), we get

$$
\frac{\partial^{2} t}{\partial x^{2}}+\frac{\partial^{2} t}{\partial y^{2}}=\frac{\partial^{2} t}{\partial r^{2}}+\frac{1}{r} \cdot \frac{\partial t}{\partial r}+\frac{1}{r^{2}} \cdot \frac{\partial^{2} t}{\partial \phi^{2}}
$$

Substituting it in eqn (2.8), we get,

$$
\left[\frac{\partial^{2} t}{\partial r^{2}}+\frac{1}{r} \cdot \frac{\partial t}{\partial r}+\frac{1}{r^{2}} \cdot \frac{\partial^{2} t}{\partial \phi^{2}}+\frac{\partial^{2} t}{\partial z^{2}}\right]+\frac{q_{g}}{k}=\frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau}
$$

which is the same as eqn. (2.22)

### 2.4. GENERAL HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATES

Consider an elemental volume having the coordinates $(r, \phi, \theta)$, for three dimensional heat conduction analysis, as shown in Fig. 2.3.


Fig. 2.3. Elemental volume for three-dimensional heat conduction analysis - Spherical coordinates.
The volume of the element $=d r \cdot r d \theta \cdot r \sin \theta d \phi$
Let,

$$
q_{g}=\text { Heat generation (uniform) per unit volume per unit time. }
$$

Further let us assume that $k$ (thermal conductivity), $\rho$ (density), $c$ (specific heat) do not alter with position.

## 30 Heat and Mass Transfer

A. Net heat accumulated in the element due to conduction of heat from all the coordinate directions considered:
Heat flow through $r-\theta$ plane; $\phi$-direction :
Heat influx, $\quad Q_{\phi}^{\prime}=-k(d r . r d \theta) \frac{\partial t}{r \cdot \sin \theta . \partial \phi} d \tau$
Heat efflux,

$$
\begin{equation*}
Q_{(\phi+d \phi)}^{\prime}=Q_{\phi}^{\prime}+\frac{\partial}{r \cdot \sin \theta \cdot \partial \phi}\left(Q_{\phi}^{\prime}\right) r \sin \theta \cdot d \phi \tag{i}
\end{equation*}
$$

$\therefore$ Heat accumulated in the element due to heat flow in the $\phi$-direction,

$$
\begin{align*}
d Q_{\phi}^{\prime} & =Q_{\phi}^{\prime}-Q^{\prime}{ }_{(\phi+\mathrm{d} \phi)} \\
& =-\frac{1}{r \sin \theta} \cdot \frac{\partial}{\partial \phi}\left(Q_{\phi}^{\prime}\right) r \sin \theta \cdot d \phi \\
& =-\frac{1}{r \sin \theta} \cdot \frac{\partial}{\partial \phi}\left[-k(d r \cdot r d \theta) \frac{1}{r \sin \theta} \cdot \frac{\partial t}{\partial \phi} \cdot d \tau\right] r \sin \theta \cdot d \phi \\
& =k(d r \cdot r d \theta \cdot r \sin \theta \cdot d \phi) \frac{1}{r^{2} \sin ^{2} \theta} \cdot \frac{\partial^{2} t}{\partial \phi^{2}} d \tau
\end{align*}
$$

Heat flow in $r-\phi$ plane, $\theta$-direction :
Heat influx,

$$
\begin{equation*}
Q_{\theta}^{\prime}=-k(d r . r \sin \theta \cdot d \phi) \frac{\partial t}{r \partial \theta} \cdot d \tau \tag{iii}
\end{equation*}
$$

Heat efflux,

$$
\begin{equation*}
Q_{(\theta+d \theta)}^{\prime}=Q_{\theta}^{\prime}+\frac{\partial}{r \partial \theta}\left(Q_{\theta}^{\prime}\right) r d \theta \tag{iv}
\end{equation*}
$$

$\therefore \quad$ Heat accumulated in the element due to heat flow in the $\theta$-direction,

$$
\begin{aligned}
d Q_{\theta}^{\prime} & =Q_{\theta}^{\prime}-Q_{(\theta+d \theta)}^{\prime} \quad \quad \text { [subtracting (iv) from (iii)] } \\
& =-\frac{\partial}{r . \partial \theta}\left(Q_{\theta}^{\prime}\right) r \cdot d \theta \\
& =-\frac{\partial}{r . \partial \theta}\left[-k(d r . r \sin \theta \cdot d \phi) \frac{\partial t}{r \cdot \partial \theta} \cdot d \tau\right] r \cdot d \theta
\end{aligned}
$$



Spherical vessels.

$$
\begin{align*}
= & \frac{k}{r} \frac{d r \cdot r d \phi \cdot r d \theta}{r} \frac{\partial}{\partial \theta}\left[\sin \theta \cdot \frac{\partial t}{\partial \theta}\right] d \tau \\
& =k(d r \cdot r d \theta \cdot r \sin \theta \cdot d \phi) \frac{1}{r^{2} \sin \theta} \cdot \frac{\partial}{\partial \theta}\left[\sin \theta \cdot \frac{\partial t}{\partial \theta}\right] d \tau \tag{2.26}
\end{align*}
$$

Heat flow in $\theta-\phi$ plane, $r$-direction :
Heat influx, $\quad Q_{r}^{\prime}=-k(r d \theta \cdot r \sin \theta \cdot d \phi) \frac{\partial t}{\partial r} \cdot \partial \tau$
Heat efflux, $\quad Q_{(r+d r)}^{\prime}=Q_{r}^{\prime}+\frac{\partial}{\partial r}\left(Q_{r}^{\prime}\right) d r$
$\therefore \quad$ Heat accumulation in the element due to heat flow in the $r$-direction,

$$
\begin{align*}
d Q_{r}^{\prime} & =Q_{r}^{\prime}-Q_{(r+d r)}^{\prime} \\
& =-\frac{\partial}{\partial r}\left(Q_{r}^{\prime}\right) d r \\
& =-\frac{\partial}{\partial r}\left[-k(r d \theta \cdot r \sin \theta \cdot d \phi) \frac{\partial t}{\partial r} \cdot d \tau\right] d r \\
& =k d \theta \cdot \sin \theta \cdot d \phi d r \frac{\partial}{\partial r}\left[r^{2} \cdot \frac{\partial t}{\partial r}\right] d \tau \\
& =k(d r \cdot r d \theta \cdot r \sin \theta \cdot d \phi) \frac{1}{r^{2}} \cdot \frac{\partial}{\partial r}\left[r^{2} \cdot \frac{\partial t}{\partial r}\right] d \tau \tag{2.27}
\end{align*}
$$

$$
\text { [subtracting }(v i) \text { from }(v)]
$$

Net heat accumulated in the element

$$
\begin{equation*}
=k d r \cdot r d \theta \cdot r \sin \theta \cdot d \phi\left[\frac{1}{r^{2} \sin ^{2} \theta} \cdot \frac{\partial^{2} t}{\partial \phi^{2}}+\frac{1}{r^{2} \sin \theta} \cdot \frac{\partial}{\partial \theta}\left(\sin \theta \cdot \frac{\partial t}{\partial \theta}\right)+\frac{1}{r^{2}} \cdot \frac{\partial}{\partial r}\left(r^{2} \cdot \frac{\partial t}{\partial r}\right)\right] d \tau \tag{2.28}
\end{equation*}
$$

B. Heat generated within the element $\left(Q_{g}^{\prime}\right)$ :

The total heat generated within the element is given by,

$$
\begin{equation*}
Q_{g}^{\prime}=q_{g}(d r . r d \theta \cdot r \sin \theta \cdot d \phi) d \tau \tag{2.29}
\end{equation*}
$$

C. Energy stored in the element :

The increase in thermal energy in the element is equal to

$$
\begin{equation*}
\rho(d r . r d \theta \cdot r \sin \theta \cdot d \phi) c \cdot \frac{\partial t}{\partial \tau} \cdot d \tau \tag{2.30}
\end{equation*}
$$

Now,

$$
(A)+(B)=(C)
$$

...Energy balance/equation

$$
\begin{aligned}
& \therefore \quad k d r \cdot r d \theta \cdot r \sin \theta \cdot d \phi\left[\frac{1}{r^{2} \sin ^{2} \theta} \cdot \frac{\partial^{2} t}{\partial \phi^{2}}+\frac{1}{r^{2} \sin \theta} \cdot \frac{\partial}{\partial \theta}\left(\sin \theta \cdot \frac{\partial t}{\partial \theta}\right)+\frac{1}{r^{2}} \cdot \frac{\partial}{\partial r}\left(r^{2} \cdot \frac{\partial t}{\partial r}\right)\right] \cdot d \tau \\
&+q_{g}(d r \cdot r d \theta \cdot r \sin \theta \cdot d \phi) d \tau=\rho(d r \cdot r d \theta \cdot r \sin \theta \cdot d \phi) c \cdot \frac{\partial t}{\partial \tau} \cdot d \tau
\end{aligned}
$$

Dividing both sides by $k .(d r . r d \theta . r \sin \theta \cdot d \phi) d \tau$, we get

$$
\begin{align*}
& {\left[\frac{1}{r^{2} \sin ^{2} \theta} \cdot \frac{\partial^{2} t}{\partial \phi^{2}}+\frac{1}{r^{2} \sin \theta} \cdot \frac{\partial}{\partial \theta}\left(\sin \theta \cdot \frac{\partial t}{\partial \theta}\right)+\frac{1}{r^{2}} \cdot \frac{\partial}{\partial r}\left(r^{2} \cdot \frac{\partial t}{\partial r}\right)\right]+\frac{q_{g}}{k}} \\
& \quad=\frac{\rho c}{k} \cdot \frac{\partial t}{\partial \tau}=\frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \tag{2.31}
\end{align*}
$$

Equation (2.31) is the general heat conduction equation in spherical coordinates.
In case there are not heat sources present and the heat flow is steady and one-dimensional, then eqn. (2.31) reduces to

$$
\begin{equation*}
\frac{1}{r^{2}} \cdot \frac{d}{d r}\left(r^{2} \cdot \frac{d t}{d r}\right)=0 \tag{2.32}
\end{equation*}
$$

## 30 Heat and Mass Transfer

Equation (2.31) can also be derived by transformation of coordinates as follows :

$$
x=r \sin \theta \sin \phi ; y=r \sin \theta \cos \phi ; z=r \cos \theta
$$

### 2.5. HEAT CONDUCTION THROUGH PLANE AND COMPOSITE WALLS

### 2.5.1. HEAT CONDUCTION THROUGH A PLANE WALL

## Case I : Uniform thermal conductivity

Refer to Fig. 2.4 (a) Consider a plane wall of homogeneous material through which heat is flowing only in $x$-direction.

Let,

$$
\begin{aligned}
& L= \text { Thickness of the plane wall, }, \\
& A= \text { Cross-sectional area of the } \\
& \text { wall, } \\
& k= \text { Thermal conductivity of the } \\
& \text { wall material, and } \\
& t_{1}, t_{2}= \text { Temperatures maintained at } \\
& \text { the two faces } 1 \text { and } 2 \text { of the } \\
& \text { wall, respectively. }
\end{aligned}
$$

The general heat conduction equation in cartesian coordinates is given by


$$
\begin{equation*}
\frac{\partial^{2} t}{d x^{2}}+\frac{\partial^{2} t}{d y^{2}}+\frac{\partial^{2} t}{d z^{2}}+\frac{q_{g}}{k}=\frac{1}{\alpha} \cdot \frac{\partial t}{\partial \tau} \tag{Eqn.2.8}
\end{equation*}
$$

If the heat conduction takes place under the conditions, steady state $\left(\frac{\partial t}{\partial \tau}=0\right)$, one-dimensional

(a)

Fig. 2.4. Heat conduction through a plane wall. $\left[\frac{\partial^{2} t}{\partial y^{2}}=\frac{\partial^{2} t}{\partial z^{2}}=0\right]$ and with no internal heat generation $\left(\frac{q_{g}}{k}=0\right)$ then the above equation is reduced to

$$
\begin{equation*}
\frac{\partial^{2} t}{d x^{2}}=0, \quad \text { or } \quad \frac{d^{2} t}{d x^{2}}=0 \tag{2.33}
\end{equation*}
$$

By integrating the above differential twice, we have

$$
\begin{equation*}
\frac{d t}{d x}=C_{1} \quad \text { and } \quad t=C_{1} x+C_{2} \tag{2.34}
\end{equation*}
$$

where $C_{1}$ and $C_{2}$ are the arbitrary constants. The values of these constants may be calculated from the known boundary conditions as follows :

$$
\begin{array}{lll}
\text { At } & x=0 & t=t_{1} \\
\text { At } & x=L & t=t_{2}
\end{array}
$$

Substituting the values in the eqn. (2.34), we get

$$
t_{1}=O+C_{2} \quad \text { and } \quad t_{2}=C_{1} L+C_{2}
$$

After simplification, we have, $C_{2}=t_{1}$ and $C_{1}=\frac{t_{2}-t_{1}}{L}$
Thus, the eqn. (2.34) reduces to :

$$
\begin{equation*}
t=\left(\frac{t_{2}-t_{1}}{L}\right) x+t_{1} \tag{2.35}
\end{equation*}
$$

The eqn. (2.35) indicates that temperature distribution across a wall is linear and is independent of thermal conductivity. Now heat through the plane wall can be found by using Fourier's equation as follows :

$$
Q=-k A \frac{d t}{d x} \quad\left(\text { where, } \quad \frac{d t}{d x}=\text { Temperature gradient }\right)
$$

[Eqn.(1.1)]
But, $\quad \frac{d t}{d x}=\frac{d}{d x}\left[\left(\frac{t_{2}-t_{1}}{L}\right) x+t_{1}\right]=\frac{t_{2}-t_{1}}{L}$

$$
\begin{equation*}
\therefore \quad Q=-k A \frac{\left(t_{2}-t_{1}\right)}{L}=\frac{k A\left(t_{1}-t_{2}\right)}{L} \tag{2.36}
\end{equation*}
$$

Eqn (2.36) can be written as :

$$
\begin{equation*}
Q=\frac{\left(t_{1}-t_{2}\right)}{(L / k A)}=\frac{\left(t_{1}-t_{2}\right)}{\left(R_{\mathrm{th}}\right)_{\mathrm{cond}} .} \tag{2.37}
\end{equation*}
$$

where, $\quad\left(R_{t h}\right)_{\text {cond. }}=$ Thermal resistance to heat conduction. Fig. $2.4(b)$ shows the equivalent thermal circuit for heat flow through the plane wall.
Let us now find out the condition when instead of space, weight is the main criterion for selection of the insulation of a plane wall.

Thermal resistance (conduction) of the wall, $\left(R_{t h}\right)_{\text {cond. }}=\frac{L}{k A}$
Weight of the wall, $W=\rho A L$
Eliminating $L$ from (i) and (ii), we get

$$
W=\rho \mathrm{A} \cdot\left(R_{t h}\right)_{\text {cond. }} k A=(\rho \cdot k) A^{2} \cdot\left(R_{t h}\right)_{\text {cond. }}
$$

The eqn., (2.38) stipulates the condition that, for a specified thermal resistance, the lightest insulation will be one which has the smallest product of density $(\rho)$ and thermal conductivity $(k)$.

## Case II. Variable thermal conductivity

A. Temperature variation in terms of surface temperatures $\left(t_{1}, t_{2}\right)$ :


A diesel engine is more efficient due to internal combustion and better heat.

But,

$$
\begin{equation*}
Q=k_{m} A\left(\frac{t_{1}-t_{2}}{L}\right) \tag{2.43}
\end{equation*}
$$

Equating these eqns. (2.47) and (2.43), we have

$$
\begin{align*}
k_{m} & =\frac{1}{\left(t_{1}-t_{2}\right)}\left[\int_{t_{1}}^{t_{2}}\left[k_{0} f(t) d t\right]\right. \\
& =\frac{1}{\left(t_{1}-t_{2}\right)} \int_{t_{2}}^{t_{1}}\left[k_{0} f(t) d t\right] \tag{2.48}
\end{align*}
$$

The effect of $+\beta$ and $-\beta$ on temperature is depicted in Fig. 2.5.

### 2.5.2. HEAT CONDUCTION THROUGH A COMPOSITE WALL

Refer to Fig. 2.6 (a). Consider the transmission of heat through a composite wall consisting of a number of slabs.

Let, $\quad L_{A}, L_{B}, L_{C}=$ Thicknesses of slabs $A, B$ and $C$ respectively (also called path lengths), $k_{A}, k_{B}, k_{C}=$ Thermal conductivities of the slabs $A, B$, and $C$ respectively, $t_{1}, t_{4}\left(t_{1}>t_{4}\right)=$ Temperatures at the wall surfaces 1 and 4 respectively, and

$$
t_{2}, t_{3}=\text { Temperatures at the interfaces } 2 \text { and } 3 \text { respectively. }
$$

Since the quantity of heat transmitted per unit time through each slab/layer is same, we have,

$$
Q=\frac{k_{A} \cdot A\left(t_{1}-t_{2}\right)}{L_{A}}=\frac{k_{B} \cdot A\left(t_{2}-t_{3}\right)}{L_{B}}=\frac{k_{C} \cdot A\left(t_{3}-t_{4}\right)}{L_{C}}
$$

(Assuming that there is a perfect contact between the layers and no temperature drop occurs across the interface between the materials).

(a)


$$
\mathrm{R}_{\mathrm{th} . \mathrm{A}}=\frac{L_{A}}{k_{A} \cdot A}, \quad \mathrm{R}_{\mathrm{th} \cdot \mathrm{~B}}=\frac{L_{B}}{k_{B} \cdot A}, \quad \mathrm{R}_{\mathrm{th.C}}=\frac{L_{C}}{k_{C} \cdot A}
$$

(b)

Fig. 2.6. Steady state conduction through a composite wall.

Rearranging the above expression, we get

$$
\begin{align*}
t_{1}-t_{2} & =\frac{Q \cdot L_{A}}{k_{A} \cdot A}  \tag{i}\\
t_{2}-t_{3} & =\frac{Q \cdot L_{B}}{k_{B} \cdot A}  \tag{ii}\\
t_{3}-t_{4} & =\frac{Q \cdot L_{c}}{k_{C} \cdot A} \tag{iii}
\end{align*}
$$

Adding (i), (ii) and (iii), we have

$$
\left(t_{1}-t_{4}\right)=Q\left[\frac{L_{A}}{k_{A} \cdot A}+\frac{L_{B}}{k_{B} \cdot A}+\frac{L_{C}}{k_{C} \cdot A}\right]
$$

or,

$$
\begin{equation*}
Q=\frac{A\left(t_{1}-t_{4}\right)}{\left[\frac{L_{A}}{k_{A}}+\frac{L_{B}}{k_{B}}+\frac{L_{C}}{k_{C}}\right]} \tag{299}
\end{equation*}
$$

or,

$$
\begin{equation*}
Q=\frac{\left(t_{1}-t_{4}\right)}{\left[\frac{L_{A}}{k_{A} \cdot A}+\frac{L_{B}}{k_{B} \cdot A}+\frac{L_{C}}{k_{C} \cdot A}\right]}=\frac{\left(t_{1}-t_{4}\right)}{\left[R_{t h-A}+R_{t h-B}+R_{t h-C}\right]} \tag{a}
\end{equation*}
$$

If the composite wall consists of $n$ slabs/layers, then

$$
\begin{equation*}
Q=\frac{\left[t_{1}-t_{(n+1)}\right]}{\sum_{1}^{n} \frac{L}{k A}} \tag{2.50}
\end{equation*}
$$

In order to solve more complex problems involving both series and parallel thermal resistances, the electrical analogy may be used. A typical problem and its analogous electric circuit are shown in Fig. 2.7.

$$
\begin{equation*}
Q=\frac{\Delta t_{\text {overall }}}{\sum R_{t h}} \tag{2.51}
\end{equation*}
$$



Fig. 2.7. Series and parallel one-dimensional heat transfer through a composite wall and electrical analog.

For the proper design of fins, the knowledge of temperature distribution along the fin is necessary. In this article the mathematical analysis for finding out the temperature distribution and heat flow from different types of fins is dealt with.

The following assumptions are made for the analysis of heat flow through the fin :

1. Steady state heat conduction.
2. No heat generation within the fin.
3. Uniform heat transfer coefficient ( $h$ ) over the entire surface of the fin.
4. Homogeneous and isotropic fin material (i.e. thermal conductivity of material constant).
5. Negligible contact thermal resistance.
6. Heat conduction one-dimensional.
7. Negligible radiation.

### 2.10.2. HEAT FLOW THROUGH "RECTANGULAR FIN"

Consider a rectangular fin protruding from a wall surface as shown in Fig. 2.121.


Fig. 2.121. Rectangular fin of uniform cross-section.
Let, $\quad l=$ Length of the fin (perpendicular to surface from which heat is to be removed),
$b=$ Width of the fin (parallel to the surface from which heat is to be removed),
$y=$ Thickness of the fin,
$P=$ Perimeter of the fin $[=2(b+y)]$,
$A_{c s}=$ Area of cross-section (=by),
$t_{o}=$ Temperature at the base of the fin, and
$t_{a}=$ Temperature of the ambient/surrounding fluid,
$k=$ Thermal conductivity (constant), and
$h=$ Heat transfer coefficient (convective).
In order to determine the governing differential equation for the fins, shown in Fig. 2.121, consider the heat flow to and from an element $d x$ thick at a distance $x$ from the base.

Heat conducted into the element at plane $x$,

$$
\begin{equation*}
Q_{x}=-k A_{c s}\left[\frac{d t}{d x}\right]_{x} \tag{i}
\end{equation*}
$$

Heat conducted out of the element at plane $(x+d x)$

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$$
\begin{equation*}
Q_{(x+d x)}=-k A_{c s}\left[\frac{d t}{d x}\right]_{x+d x} \tag{ii}
\end{equation*}
$$

Heat convected out of the element between the planes $x$ and $(x+d x)$,

$$
Q_{c o n v}=h(P . d x)\left(t-t_{a}\right)
$$

Applying an energy balance on the element, we can write

$$
\begin{align*}
Q_{x} & =Q_{(x+d x)}+Q_{\text {conv. }} \\
-k A_{c s}\left[\frac{d t}{d x}\right]_{x} & =-k A_{c s}\left[\frac{d t}{d x}\right]_{x+d x}+h(P . d x)\left(t-t_{a}\right) \tag{2.128}
\end{align*}
$$

Making a Taylor's expansion of the temperature gradient at $(x+d x)$ in terms of that at $x$, we get

$$
\left(\frac{d t}{d x}\right)_{x+d x}=\left(\frac{d t}{d x}\right)_{x}+\frac{d}{d x}\left(\frac{d t}{d x}\right)_{x} d x+\frac{d^{2}}{d x^{2}}\left(\frac{d t}{d x}\right) \frac{(d x)^{2}}{2!}+\ldots
$$

Substituting this in eqn. (2.128), we have

$$
-k A_{c s}\left[\frac{d t}{d x}\right]_{x}=-k A_{c s}\left[\frac{d t}{d x}\right]_{x}-k A_{C S}\left[\frac{d^{2} t}{d x^{2}}\right]_{x} d x-k A_{c s}\left[\frac{d^{3} t}{d x^{3}}\right] \frac{(d x)^{2}}{2!}+. .+h(P . d x)\left(t-t_{a}\right)
$$

Neglecting higher terms as $d x \rightarrow 0$, we have

$$
\begin{gathered}
-k A_{c s}\left[\frac{d t}{d x}\right]=-k A_{c s}\left[\frac{d t}{d x}\right]-k A_{c s}\left[\frac{d^{2} t}{d x^{2}}\right] d x+h(P \cdot d x)\left(t-t_{a}\right) \\
k A_{c s}\left[\frac{d^{2} t}{d x^{2}}\right] d x-h(P . d x)\left(t-t_{a}\right)=0
\end{gathered}
$$

Dividing both sides by $A_{c s} d x$, we get,
or,

$$
\begin{align*}
& k \frac{d^{2} t}{d x^{2}}-\frac{h P}{A_{c s}}\left(t-t_{a}\right)=0 \\
& \frac{d^{2} t}{d x^{2}}-\frac{h P}{k A_{c s}}\left(t-t_{a}\right)=0 \tag{2.129}
\end{align*}
$$

Eqn. (2.129) is further simplified by transforming the dependent variable by defining the temperature excess $\theta$ as,

$$
\left.\theta_{(x)}=t_{(x)}\right)-t_{(a)}
$$

As the ambient temperature $t_{a}$ is constant, we get by differentiation

$$
\begin{equation*}
\frac{d \theta}{d x}=\frac{d t}{d x} ; \quad \frac{d^{2} \theta}{d x^{2}}=\frac{d^{2} t}{d x^{2}} \tag{2.130}
\end{equation*}
$$

Thus, $\frac{d^{2} \theta}{d x^{2}}-m^{2} \theta=0$
where $\quad m=\sqrt{\frac{h P}{k A_{c s}}}$
Eqns. (2.129) and (2.130) represent a general form of the energy equation for one-dimensional heat dissipation from an extended surface (fin). The parameter $m$, for a given fin, is constant provided the convective film coefficient $h$ is constant over the whole surface and the thermal conductivity $k$ is constant within the temperature range considered. Then the general solution of this linear and homogeneous second order differential equation is of the form :

$$
\begin{align*}
\theta & =\mathrm{C}_{1} e^{m x}+C_{2} e^{-m x}  \tag{2.131}\\
\text { or, } \quad\left[t-t_{a}\right. & \left.=C_{1} e^{m x}+C_{2} e^{-m x}\right]
\end{align*}
$$

### 2.10.2.4. Efficiency and effectiveness of fin

## Efficiency of fin $\left(\boldsymbol{\eta}_{f i n}\right)$ :

The efficiency of a fin is defined as the ratio of the actual heat transferred by the fin to the maximum heat transferable by fin, if entire fin area were at base temperature.
i.e.,

$$
\eta_{f i n}=\frac{\text { Actual heat transferred by the fin }\left(Q_{f i n}\right)}{\begin{array}{l}
\text { Maximum heat that would be transferred if whole surface of the fin } \\
\text { is maintained at the base temperature }\left(Q_{\max }\right)
\end{array}}
$$

For a fin which is infinitely long (Art. 2.10.2.1)

$$
\begin{equation*}
\eta_{f i n}=\frac{\sqrt{P h k A_{c s}\left(t_{o}-t_{a}\right)}}{h P l\left(t_{o}-t_{a}\right)}=\sqrt{\frac{k A_{c s}}{h P l^{2}}}=\frac{1}{m l} \tag{2.138}
\end{equation*}
$$

For a fin which is insulated at the tip (Art. 2.10.2.2) :
$\begin{aligned} \quad \eta_{f i n} & =\frac{\sqrt{P h k A_{c s}}\left(t_{o}-t_{a}\right) \tanh (m l)}{h P l\left(t_{o}-t_{a}\right)}=\frac{\tanh (m l)}{m l} \\ \text { where, } \quad m l & =\sqrt{\frac{h P}{k A_{c s}}} \cdot l=\sqrt{\frac{h(2 b+2 y)}{k b y}} \cdot l\end{aligned} \quad[\because P=2(b+y)]$
Now, if the fin is sufficiently wide then the term $2 b$ will be large compared to $2 y$, then

$$
m l=\sqrt{\frac{2 h b}{k b y}} \cdot l=\sqrt{\frac{2 h}{k y}} \cdot l=\sqrt{\frac{2 h}{k y}} \cdot l^{3 / 2}=\sqrt{\frac{2 h}{k A_{p}}} \cdot l^{3 / 2}
$$

where,

$$
y . l=A_{p}=\text { Profile area of the fin. }
$$

Thus the fin efficiency is a function of $m l$ or $\sqrt{\frac{2 h}{k A_{p}}} \cdot l^{3 / 2}$.
The efficiency of a real rectangular fin which is long, wide and thin can be calculated by replacing $l$ by a corrected length $l_{c}$, given by

$$
\begin{equation*}
l_{c}=1+\frac{y}{2} \tag{2.140}
\end{equation*}
$$

This corrected length compensates for the fact that there is connective heat loss from the tip of a real fin. The efficiency is then written as

$$
\eta_{f i n}=\frac{\tanh \left[\sqrt{2 h / k y}\left(l+\frac{y}{2}\right)\right]}{\sqrt{2 h / k y}\left[l+\begin{array}{l}
y  \tag{2.141}\\
2
\end{array}\right]}
$$

The heat flow becomes,

$$
\begin{align*}
Q & =\sqrt{h P k A_{c s}}\left(t_{o}-t_{a}\right) \tanh \left[\sqrt{2 h / k y}\left(l+\frac{y}{2}\right)\right] \\
& =\eta_{f i n} \sqrt{h P l_{c}}\left(t_{o}-t_{a}\right) \tag{2.142}
\end{align*}
$$

The efficiency of a fin forms a criterion for judging the relative merits of fins of different geometrics or materials.

## Effectiveness of fin $\left(\boldsymbol{\varepsilon}_{\text {fin }}\right)$ :

Effectiveness of fin is the ratio of the fin heat transfer rate to the heat transfer rate that would exist without a fin.

$$
\begin{equation*}
\varepsilon_{\text {fin }}=\frac{Q_{\text {with fin }}}{Q_{\text {without fin }}}=\frac{\sqrt{P h k A_{c s}}\left(t_{o}-t_{a}\right)}{h A_{c s}\left(t_{o}-t_{a}\right)}=\sqrt{\frac{P k}{h A_{c s}}} \tag{2.143}
\end{equation*}
$$

(...in case of infinitely long fin.)

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For a straight rectangular fin of thickness of $y$ and width $b$,

$$
\begin{array}{ll} 
& \frac{P}{A_{c s}}=\frac{2(b+y)}{b \cdot y} \simeq \frac{2}{y} \\
\therefore \quad & \varepsilon_{f i n}=\sqrt{\frac{2 k}{h y}} \tag{2.144}
\end{array}
$$

From the relation for fin effectiveness, following results can be inferred :

1. Fin effectiveness $\sqrt{\frac{P k}{h A_{c s}}}$ should be greater than unity if the rate of heat transfer from the primary surface is to be improved. It has been observed that use of fins on surfaces is justified only if $\frac{P k}{h A_{c s}}>5$.
2. If the ratio of $P$ (perimeter) and $A_{c s}$ (cross-sectional area) is increased the effectiveness of fin is improved. Due to this reason, thin and closely spaced fins are preferred; the lower limit on the distance between two adjacent fins (pitch) is governed by the thickness of boundary layer that develops on the surface of the fin.
3. Use of fins is only justified where $h$ is small; finning is hardly justified unless $h<0.25\left[\frac{k P}{A}\right]$. If the value of $h$ is large (as experienced in boiling, condensation and high velocity fluids), the fins may actually produce a reduction is heat transfer.
4. It is also apparent that the use of fins will be more effective with materials of large thermal conductivities [Although copper is superior to aluminium regarding thermal conductivity, yet fins are generally made of aluminium since it (aluminium) is cheaper in cost and lighter in weight].
Relation between $\eta_{\text {fin }}$ and $\varepsilon_{f i n}$ :
The performance parameters (i.e. $\eta_{f i n}$ and $\varepsilon_{f i n}$ ), in case of a fin insulated at the tip, are related to each other by the following expressions :

Efficiency of fin,

$$
\begin{align*}
& \eta_{f i n}=\frac{\sqrt{P h k A_{c s}}\left(t_{0}-t_{a}\right) \tanh (m l)}{h P l\left(t_{0}-t_{a}\right)}  \tag{i}\\
& \varepsilon_{f i n}=\frac{\sqrt{P h k A_{c s}}\left(t_{0}-t_{a}\right) \tanh (m l)}{h A_{c s}\left(t_{o}-t_{a}\right)} \tag{ii}
\end{align*}
$$

Dividing eqn. (ii) by eqn. (i), we have

$$
\begin{align*}
\frac{\varepsilon_{f i n}}{\eta_{f i n}} & =\frac{P l}{A_{c s}}  \tag{2.145}\\
\varepsilon_{f i n}=\eta_{f i n} \frac{P l}{A_{c s}} & =\eta_{f i n} \times \frac{\text { Surface area of the fin }}{\text { Cross-sectional area of the fin }}
\end{align*}
$$

or,
It is evident from the above equations that an increase in fin effectiveness can be obtained by increasing the length of the fin but it decreases the efficiency of the fin on the other hand.

Example 2.126. A longitudinal copper fin $\left(k=380 \mathrm{~W} / \mathrm{m}^{\circ} \mathrm{C}\right) 600 \mathrm{~mm}$ long and 5 mm diameter is exposed to air stream at $20^{\circ} \mathrm{C}$. The convective heat transfer coefficient is $20 \mathrm{~W} / \mathrm{m}^{2}{ }^{\circ} \mathrm{C}$. If the fin base temperature is $150^{\circ} \mathrm{C}$, determine :
(i) The heat transferred, and
(ii) The efficiency of the fin.
[P.U., 1997]

From the properties of Bessel's functions, we know that

$$
\frac{d\left[I_{n}(z)\right]}{d x}=I_{(n+1)}(z) \frac{d}{d x}(z)
$$

so that for $n=0$

$$
\begin{align*}
& \quad \frac{d\left[I_{0}(2 B \sqrt{x})\right]}{d x}=\left[I_{1}(2 B \sqrt{x})\right] B x^{-1 / 2} \\
& \therefore \\
& \text { or, } \quad Q=\frac{k b y \theta_{o}}{I_{0}(2 B \sqrt{l})}\left[\frac{d}{d x}\left\{I_{0}(2 B \sqrt{x})\right\}\right]_{x=l} \\
& \text { or, } \quad Q=\frac{k b y \theta_{o}}{I_{0}(2 B \sqrt{l})}\left[I_{1}(2 B \sqrt{x}) B x^{-1 / 2}\right]_{x=l} \\
& \text { or, } \quad Q=\frac{k b y \theta_{o}}{I_{0}(2 B \sqrt{l})}\left[I_{1}(2 B \sqrt{l}) \times \frac{\theta_{o} l}{I_{0}}\right.  \tag{2.162}\\
& \text { or, } \quad Q=\frac{k B b y \theta_{o}}{\sqrt{l}} \times \frac{I_{1}(2 B \sqrt{l})}{I_{0}(2 B \sqrt{l})} \\
& \text { But } \quad B=\sqrt{\frac{2 h l}{k y}} \\
& \therefore \quad Q=\frac{k b y \theta_{o}}{\sqrt{l}} \times \sqrt{\frac{2 h l}{k y}} \times \frac{I_{1}(2 B \sqrt{l})}{I_{0}(2 B \sqrt{l})}  \tag{2.163}\\
& \text { or, } Q_{f i n}=b \sqrt{2 h k y} \cdot \theta_{o} \frac{I_{1}(2 B \sqrt{l})}{I_{0}(2 B \sqrt{l})} \\
& \hline
\end{align*}
$$

$$
\therefore \quad Q=k \times(b \times y) \times \frac{d}{d x}\left[\frac{\theta_{o} I_{0}(2 B \sqrt{x})}{I_{0}(2 B \sqrt{l})}\right]_{x=l} \quad \text { [using eqn. (2.161)] }
$$



Parabolic fins.

Note: The straight fins can be of rectangular, triangular and parabolic profiles; parabalic fins are the most effective but are difficult to manufacture.

### 2.10.4. ESTIMATION OF ERROR IN TEMPERATURE MEASUREMENT IN A THERMOMETER WELL

For estimating error in the value of temperature measured by a thermometer dipped in a thermometer well, the theory of extended surfaces is very helpful. A thermometer well is defined as a small tube welded radially into a pipeline through which a fluid whose temperature is to be measured is flowing.

Refer to Fig. 2.139.
Let, $\quad l=$ Length of the well/tube,
$d=$ Internal diameter of the well/tube,
$\delta=$ Thickness of well/tube,
$t_{f}=$ Temperature of the fluid flowing through the pipe (which is to be measured), and
$t_{o}=$ Temperature of the pipe-wall.
When the temperature of the fluid flowing through the pipeline is higher than the ambient temperature, the heat flows from the fluid towards the tube walls along the well. Consequently the temperature at the bottom of well becomes colder than the fluid flowing around, obviously the

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temperature shown by the thermometer will not be the true temperature of the fluid. This error may be calculated by assuming the well to be a spine protruding from the wall of a pipe in which fluid is flowing. It may be assumed, for simplicity, that there is no flow of heat from the tip of the well (i.e., the tip of the well is insulated). The temperature distribution at any distance $x$ measured from pipe wall along the temperature well is given by

$$
\begin{equation*}
\frac{\theta_{x}}{\theta_{o}}=\frac{t_{x}-t_{f}}{t_{o}-t_{f}}=\frac{\cosh [m(l-x)]}{\cosh (m l)} \tag{2.134}
\end{equation*}
$$



Fig. 2.139. Thermometer well.


Fig. 2.140. Temperature varitation in well.

At $x=l$, we have

$$
\begin{equation*}
\frac{t_{l}-t_{f}}{t_{o}-t_{f}}=\frac{\cosh [m(l-l)]}{\cosh (m l)}=\frac{1}{\cosh (m l)} \tag{2.164}
\end{equation*}
$$

[Thermometeric error]
(where, $t_{l}=$ Temperature recorded by the thermometer at the bottom of the well.)
Now, perimeter of the well, $\quad P=\pi(d+2 \delta) \simeq \pi d$,
and cross-sectional area, $A_{c s}=\pi d \delta$
$\therefore \quad \frac{P}{A_{c s}}=\frac{\pi d}{\pi d \delta}=\frac{1}{\delta}$
Then,

$$
m=\sqrt{\frac{h P}{k A_{c s}}}=\sqrt{\frac{h}{k \delta}}
$$

Thus, the temperature measured by the thermometer is not affected by the diameter of the well.
From the Eqn. (2.164) it is obvious that in order to reduce the temperature measurement error, $m l$ should be large necessitating the following :
(i) Large value of $h$ (heat transfer coefficient).
(ii) Small value of of $k$ (thermal conductivity).
(iii) Long and thin well, the pocket (protruding small tube) may be placed obliquely/inclined, if necessary, to provide a longer insertion of thermometer.

Example 2.130. A mercury thermometer placed in oil well is required to measure temperature of compressed air flowing in a pipe. The well is 140 mm long and is made of steel ( $k=50 \mathrm{~W} / \mathrm{m}^{\circ} \mathrm{C}$ ) of 1 mm thickness. The temperature recorded by the well is $100^{\circ} \mathrm{C}$ while pipe wall temperature is $50^{\circ} \mathrm{C}$. Heat transfer coefficient between the air and well wall is $30 \mathrm{~W} / \mathrm{m}^{2}{ }^{\circ} \mathrm{C}$. Estimate true temperature of air.
(M.U.)

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Let us consider a body whose initial temperature is $t_{i}$ throughout and which is placed suddenly in ambient air or any liquid at a constant temperature $t_{a}$ as shown in Fig. 4.1(a). The transient response of the body can be determined by relating its rate of change of internal energy with convective exchange at the surface. That is:


Fig. 4.1. Lumped heat capacity system.

$$
\begin{equation*}
Q=-\rho V c \frac{d t}{d \tau}=h A_{s}\left(t-t_{a}\right) \tag{4.1}
\end{equation*}
$$

where,

$$
\begin{aligned}
\rho & =\text { Density of solid, } \mathrm{kg} / \mathrm{m}^{3}, \\
V & =\text { Volume of the body, } \mathrm{m}^{3}, \\
c & =\text { Specific heat of body, } \mathrm{J} / \mathrm{kg}{ }^{\circ} \mathrm{C}, \\
h & =\text { Unit surface conductance, } \mathrm{W} / \mathrm{m}^{2 \circ} \mathrm{C}, \\
t & =\text { Temperature of the body at any time, }{ }^{\circ} \mathrm{C}, \\
A_{s} & =\text { Surface area of the body }, \mathrm{m}^{2}, \\
t_{a} & =\text { Ambient temperature, }{ }^{\circ} \mathrm{C}, \text { and } \\
\tau & =\text { Time, s. }
\end{aligned}
$$

After rearranging the eqn. (4.1), and integrating, we get

$$
\left.\begin{array}{rl} 
& \int \frac{d t}{\left(t-t_{a}\right)}
\end{array}=-\frac{h A_{s}}{\rho V c} \int d \tau\right]
$$

The boundary conditions are:
At $\tau=0, \quad t=t_{i}$ (initial surface temperature)

$$
\begin{equation*}
\therefore \quad C_{1}=\ln \left(t_{i}-t_{a}\right) \tag{4.3}
\end{equation*}
$$

Hence

$$
\ln \left(t-t_{a}\right)=-\frac{h A_{s}}{\rho V c} \tau+\ln \left(t_{i}-t_{a}\right)
$$

[Substituting the values in eqn. (4.3)]
or, $\quad \frac{t-t_{a}}{t_{i}-t_{a}}=\frac{\theta}{\theta_{i}}=\exp \left[-\frac{h A_{s}}{\rho V c} \tau\right]$

Following points are worth noting:

1. Eqn. (4.4) gives the temperature distribution in the body for Newtonian heating or cooling and it indicates that temperature rises exponentially with time as shown in Fig. 4.2.


Fig. 4.2. Newtonian heating or cooling.


Fig. 4.3. Transient temperature response.
2. The quantity $\frac{\rho V c}{h A_{s}}$ has the dimensions of time and is called thermal time constant, denoted by $\tau_{t h}$. Its value is indicative of the rate of response of a system to a sudden change in its environmental temperature i.e., how fast a body will response to a change in the environmental temperature.
where,

$$
\begin{aligned}
\tau_{t h} & =\left(\frac{1}{h A_{s}}\right)(\rho V c)=R_{t h} C_{t h} \\
R_{t h} & =\left(\frac{1}{h A_{s}}\right)=\text { Resistance to convection heat transfer, and }
\end{aligned}
$$

$$
C_{t h}\left(=\rho V_{c}\right)=\text { Lumped thermal capacitance of solid. }
$$

Fig. 4.3 shows that any increase in $R_{t h}$ or $C_{t h}$ will cause a solid to respond more slowly to changes in its thermal environmental and will increase the time required to attain the thermal equilibrium ( $\theta=0$ ).

Fig. 4.1(b) shows an analogus electric network for a lumped heat capacity system, in which $C_{t h}=\rho V c$ represents the thermal capacity of the system. The value of $C_{t h}$ can be obtained from the following thermal and electrical equations, by similarity.

$$
\begin{aligned}
Q & =(\rho V c) t=C_{t h} \cdot t \\
s & =C \cdot E
\end{aligned}
$$

...Thermal equation.
...Electrical equation.
where,

$$
\begin{aligned}
s & =\text { Capacitor charge }, \\
C & =\text { Capacitance of the condenser, and } \\
E & =\text { Voltage } .
\end{aligned}
$$

When the switch is closed [Fig. $4.1(b)$ ] the solid is charged to the temperature $\theta$. On opening the switch, the thermal energy stored as $C_{t h}$ is dissipated through the thermal resistance $R_{t h}=\left(\frac{1}{h A_{s}}\right)$ and the temperature of the body decays with time. From this analogy it is concluded that $R C$ electrical circuits may be used to determine the transient behaviour of thermal systems.

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The power on exponential, i.e., $\frac{h A_{s}}{\rho V c} \tau$ can be arranged in dimensionless form as follows.

$$
\begin{equation*}
\frac{h A_{s}}{\rho V c} \tau=\left(\frac{h V}{k A_{s}}\right)\left(\frac{A_{s}^{2} k}{\rho V^{2} c} \tau\right)=\left(\frac{h L_{c}}{k}\right)\left(\frac{\alpha \tau}{L_{c}^{2}}\right) \tag{4.5}
\end{equation*}
$$

where $\alpha=\left[\frac{k}{\rho c}\right]=$ Thermal diffusivity of the solid

$$
L_{c}=\text { Characteristic length }=\frac{\text { Volume of the solid }(V)}{\text { Surface area of the solid }\left(A_{s}\right)}
$$

The values of characteristic length $\left(L_{c}\right)$, for simple geometric shapes, are given below:
Flat plate : $L_{c}=\frac{V}{A_{s}}=\frac{L B H}{2 B H}=L / 2=$ semi-thickness
where $L, B$ and $H$ are thickness, width and height of the plate.
Cylinder (long) : $L_{c}=\frac{\pi R^{2} L}{2 \pi R L}=\frac{R}{2} \quad$ where, $\quad R=$ radius of the cylinder.
Sphere: $L_{c}=\frac{\frac{4}{3} \pi R^{3}}{4 \pi R^{2}}=\frac{R}{3} \quad$ where, $\quad R=$ radius of the sphere.
Cube: $L_{c}=\frac{L^{3}}{6 L^{2}}=\frac{L}{6} \quad$ where, $\quad L=$ Side of the cube.
Further, from eqn. (4.5):
(i) The non-dimensional factor $\frac{h L_{c}}{k}$ is called the Biot member $\boldsymbol{B}_{\boldsymbol{i}}$,
i.e.

$$
B_{i}=\frac{h L_{c}}{k}=\text { Biot number. }
$$

It gives an indication of the ratio of internal (conduction) resistance to surface (convection) resistance. When the value of $B_{i}$ is small, it indicates that the system has a small internal (conduction) resistance, i.e., relatively small temperature gradient or the existence of practically uniform temperature within the system. The convective resistance then predominates and the transient phenomenon is controlled by the convective heat exchange.

If $B_{i}<0.1$, the lumped heat capacity approach can be used to advantage with simple shapes such as plates, cylinders, spheres and cubes. The error associated is around $5 \%$.
(ii) The non-dimensional factor $\frac{\alpha \tau}{L_{c}^{2}}$ is called the Fourier number, $\boldsymbol{F}_{\mathbf{0}}$.
i.e. $\quad F_{0}=\frac{\alpha \tau}{L_{c}^{2}}=$ Fourier number

It signifies the degree of penetration of heating or cooling effect through a solid.
Using non-dimensional terms, eqn. (4.4) takes the form of

$$
\frac{\theta}{\theta_{i}}=\frac{t-t_{a}}{t_{i}-t_{a}}=e^{-B i F_{0}}
$$

The graphical representation of eqn. (4.5) for different solids (Infinite plates, infinite cylinders and infinite square rods and cubes and spheres) is shown in Fig. 4.4.

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Likewise the mass flow entering the bottom face is $\rho v d x d \tau$ and the mass leaving the top face is $\rho\{v+(\partial v / \partial y) d y\} d x d \tau$ A mass balance on the element yields :

$$
\rho u d y d \tau+\rho v d x d \tau=\rho\left(u+\frac{\partial u}{\partial x} d x\right) d y d \tau+\rho\left(v+\frac{\partial v}{\partial y} d y\right) d x d \tau
$$

Simplification gives :

$$
\begin{equation*}
\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=0 \tag{12.3}
\end{equation*}
$$

This is the mass continuity equation for two-dimensional steady flow of an incompressible fluid. The continuity equation is a mathematical expression of the fact that flow is continuous; it has no breaks in it.

### 12.2.2. Force or momentum equation

For a two-dimensional infinitesimal control volume ( $d x \times d y \times$ unit depth) within the boundary layer region, the viscous forces acting along with the momentum of fluid entering and leaving the elementary volume have been indicated in Fig. 12.5.


Fig. 12.5. Elemental control volume and frice balanco
Momentum change : The momentum flux in $x$-direction is the product of mass flowing in $x$-direction and the $x$-component $u$ of velocity. A fluid mass enters the left face at the rate $\rho u d y$ producing an $x$-momentum influx

$$
\begin{equation*}
m_{x} u=(\rho u d y) u=\rho u^{2} d y \tag{12.4}
\end{equation*}
$$

The momentum efflux through the right face is

$$
=m_{x} u+\frac{\partial}{\partial x}\left(m_{x} u\right) d x
$$

$$
\begin{align*}
& =(\rho u d y) u+\frac{\partial}{\partial x}(\rho u d y u) d y \\
& =\rho u^{2} d y+2 \rho u \frac{\partial u}{\partial x} d x d y \tag{12.5}
\end{align*}
$$

Since we are concerned only with momentum in $x$-direction, the momentum of the fluid moving in $y$-direction is obtained by multiplying the mass moving in $y$-direction also with the $x$-component $u$ of the velocity. Therefore the momentum influx from the bottom face is

$$
\begin{equation*}
m_{v} u=(\rho v d x) u=\rho u v d x \tag{12.6}
\end{equation*}
$$

and the momentum effux from the top face is

$$
\begin{align*}
& =m_{y} u+\frac{\partial}{\partial y}\left(m_{y} u\right) d y \\
& =(\rho v d x) u+\frac{\partial}{\partial y}(\rho v d x u) d y \\
& =\rho u v d x+\rho u \frac{\partial v}{\partial y} d x d y+\rho v \frac{\partial u}{\partial y} d x d y \tag{12.7}
\end{align*}
$$

The resultant momentum change in $x$-direction is, $=$ momentum efflux from the right and top faces - momentum influx from the left and bottom faces

$$
\begin{aligned}
=\left(\rho u^{2} d y 2 \rho u \frac{\partial u}{\partial x} d x d y\right) & +\left(\rho u v d x+\rho u \frac{\partial v}{\partial y} d x d y+\rho v \frac{\partial u}{\partial y} d x d y\right)-\rho u^{2} d y-\rho u v d x \\
& =2 \rho u \frac{\partial u}{\partial x} d x d y+\rho u \frac{\partial v}{\partial y} d x d y+\rho v \frac{\partial u}{\partial y} d x d y \\
& =\rho\left(u \frac{\partial u}{\partial x}+v \frac{\partial u}{\partial y}\right) d x d y+\rho u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) d x d y
\end{aligned}
$$

For the continuity equation. $(\partial u / \partial x)+(\partial v / \partial y)=0$ and therefore the net momentum transfer in $x$-direction becomes :

$$
\begin{equation*}
=\rho\left(u \frac{\partial u}{\partial x}+v \frac{\partial u}{\partial y}\right) d x d y \tag{12.8}
\end{equation*}
$$

Viscous forces : The shearing stress due to fluid viscosity is proportional to the velocity gradient and is given by Newtons' law of viscosity.

$$
\text { Shear stress, } \quad \tau=\mu \frac{\partial u}{\partial y}
$$

where $\mu$ is the dynamic viscosity of the fluid
The shearing stress at the lower face of the control volume is $\tau=\mu(\partial u / \partial y)$ and the coresponding shearing force for the area $(d x \times 1)$ is $\mu(\partial u / \partial y) d x$. The shearing stress due to viscosity at the upper face of the control volume is $\{\tau+(\partial \tau / \partial y) d y\}$ and the corresponding shearing force for the area $(d x \times 1)$ is

$$
\left(\tau+\frac{\partial \tau}{\partial y} d y\right) d x=\left[\mu \frac{\partial u}{\partial y}+\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) d y\right] d x=\left(\mu \frac{\partial u}{\partial y}+\mu \frac{\partial^{2} u}{\partial y^{2}} d y\right) d x
$$

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Since the main stream flows in $x$-direction, the shearing force in $y$-direction can be neglected. Therefore the net viscous force in the direction of motion is :

$$
\begin{equation*}
\left(\mu \frac{\partial u}{\partial y}+\mu \frac{\partial^{2} u}{\partial y^{2}} d y\right) d x-\mu \frac{\partial u}{\partial y} d x=\mu \frac{\partial^{2} u}{\partial y^{2}} d x d y \tag{12.9}
\end{equation*}
$$

For a fixed control volume and steady flow, the Newton's second law of motion stipulates that the resultant applied $x$-force equals the net rate of $x$-momentum transfer out of the volume, i.e.,

$$
\Sigma F_{x}=(x \text {-momentum efflux })-(x \text {-momentum influx })
$$

Therefore, in the absence of any pressure and gravitational forces

$$
\begin{aligned}
\mu \frac{\partial^{2} u}{\partial y^{2}} d x d y & =\rho\left(u \frac{\partial u}{\partial x}+v \frac{\partial u}{\partial y}\right) d x d y \\
\text { or } \quad \mu \frac{\partial^{2} u}{\partial y^{2}} & =\rho\left(u \frac{\partial u}{\partial x}+v \frac{\partial u}{\partial y}\right)
\end{aligned}
$$

By substituting $\mu / \rho=v_{\text {, }}$ (the kinematic viscosity), we obtain

$$
\begin{equation*}
u \frac{\partial u}{\partial x}+v \frac{\partial u}{\partial y}=v \frac{\partial^{2} u}{\partial y^{2}} \tag{12.10}
\end{equation*}
$$

This is the force or momentum equation of the boundary layer with constant properties.

## POOL BOILING

In pool boiling, vapor is generated at a superheated wall that is small compared to the dimensions of the pool of nominally stagnant liquid in which it is immersed. The motion of the liquid is induced by the boiling process itself (analogous to single-phase natural convection at a heated wall in an unbounded fluid) and the velocities are assumed to be low. These conditions are convenient for small-scale laboratory experiments and much of the understanding of boiling, such as the basic division into nucleate, transition and film boiling and studies of bubble nucleation and motion discussed in the article on Boiling, has been derived from pool boiling experiments. However, pool boiling is unusual in industrial equipment. Even if there is no forced flow of liquid past the heated wall, confinement of the liquid and close spacing of multiple heaters, as in kettle reboilers, means that conditions are closer to Forced Convective Boiling. The heat source is often a hot fluid separated from the boiling liquid by a thin metal wall, whereas electrical resistance heating is often used in pool boiling experiments. Consequently it is important to appreciate the special conditions of pool boiling experiments and to exercise caution in transferring the information they provide to large-scale industrial systems in which flow effects are generally significant. This article reviews the techniques that are used in pool boiling experiments. Pool boiling behavior is described in more detail in the articles on Boiling, Nucleate Boiling, and Burnout (Pool Boiling).

Pool boiling can be classified according to conditions in the pool, the geometry of the heated wall and the method of heating. These conditions influence the methods used to measure the primary variables of wall superheat and heat flux that are conventionally used to present boiling heat transfer performance as a "boiling curve".

In saturated pool boiling, or bulk boiling, the pool is maintained at or slightly above the saturation temperature by interaction with the vapor bubbles rising from the superheated boiling surface. (Subsidiary heaters may be used to compensate for heat lost from the walls of the containing vessel.) The pool has a free surface at which the bubbles burst; the vapor space is usually connected to a condenser that returns liquid to the pool. The system pressure is controlled by the cooling applied to the condenser. In subcooled boiling, the pool temperature distant from the boiling surface is below the saturation temperature. There can be no escape of vapor from a
subcooled pool, unless it is very shallow, so a heat sink must be provided by cooling regions on the walls of the vessel. Alternatively, a subcooled experiment can be run for a short period without heat sink, relying on the thermal capacity of the cold pool. A subcooled pool cannot have a free surface in contact with its own pure vapor. Either the boiling vessel must be connected to a separate vessel in which the pressure is controlled, or there must be a gas space above the pool. Use of a cover gas leads to a concentration of dissolved gas which can influence boiling, particularly by improving the stability of nucleation sites and reducing the superheat required for their activation. Dissolved gas can be removed by a preliminary period of saturated boiling, either in the experimental vessel or in a separate vessel from which the experimental vessel is filled. The temperature-time-dissolved gas history can influence the subsequent boiling experiments, as described in the article on Nucleate Boiling, and may be different in industrial systems.

In both saturated and subcooled pool boiling, the operation of the heat sink requires a recirculatory flow in the pool that may interact with the boiling process in ways that depend on the geometry of the pool and of the superheated boiling surface. The shape of the vessel may be constrained by the need to observe the boiling process. Early experiments on pool boiling used heating surfaces that were thin horizontal wires of materials such as platinum, heated by the passage of direct electrical current. The electrical resistance of the wire provided a measure of its temperature, averaged over its length. Such experiments are useful to demonstrate some of the basic characteristics of boiling but they suffer from the disadvantage that the length scale of the bubbles is similar to that of the heater so that their behavior is atypical of the extensive surfaces in industrial plant. Most experiments now use larger heaters in the form of horizontal cylinders with diameters in the range 10 to 20 mm , horizontal plates of circular or rectangular shape and vertical or sloping rectangular plates, with dimensions in the range 5 to 100 mm . Heaters much larger than this are rarely used because of the large power requirements resulting from the high heat fluxes in nucleate boiling. The small heaters interact with the recirculation of liquid in the pool through edge effects or because their dimensions are comparable with the critical wavelengths of interfacial instabilities in film boiling. The recirculatory flows that must return liquid right to the wall in nucleate boiling are rarely considered, except in the special case of vertical flow counter to the vapor flow.

## Nucleation

Nucleation is typically defined to be the process that determines how long an observer has to wait before the new phase or self-organized structure appears. For example, if a volume of water is cooled (at atmospheric pressure) below $0{ }^{\circ} \mathrm{C}$, it will tend to freeze into ice, but volumes of water cooled only a few degrees below $0{ }^{\circ} \mathrm{C}$ often stay completely free of ice for long periods. At these conditions, nucleation of ice is either slow or does not occur at all. However, at lower temperatures ice crystals appear after little or no delay. At these conditions ice nucleation is fast. Nucleation is commonly how first-order phase transitions start, and then it is the start of the process of forming a new thermodynamic phase. In contrast, new phases at continuous phase transitions start to form immediately.

Nucleation is often found to be very sensitive to impurities in the system. These impurities may be too small to be seen by the naked eye, but still can control the rate of nucleation. Because of this, it is often important to distinguish between heterogeneous nucleation and homogeneous nucleation. Heterogeneous nucleation occurs at nucleation sites on surfaces in the system. Homogeneous nucleation occurs away from a surface.

The time until the appearance of the first crystal is also called primary nucleation time, to distinguish it from secondary nucleation times. Primary here refers to the first nucleus to form, while secondary nuclei are crystal nuclei produced from a preexisting crystal. Primary nucleation describes the transition to a new phase that does not rely on the new phase already being present, either because it is the very first nucleus of that phase to form, or because the nucleus forms far from any pre-existing piece of the new phase. Particularly in the study of crystallisation,
secondary nucleation can be important. This is the formation of nuclei of a new crystal directly caused by pre-existing crystals.

For example, if the crystals are in a solution and the system is subject to shearing forces, small crystal nuclei could be sheared off a growing crystal, thus increasing the number of crystals in the system. So both primary and secondary nucleation increase the number of crystals in the system but their mechanisms are very different, and secondary nucleation relies on crystals already being present.

## Laminar and Turbulent Boundary Layers

A boundary layer may be laminar or turbulent. A laminar boundary layer is one where the flow takes place in layers, i.e., each layer slides past the adjacent layers. This is in contrast to Turbulent Boundary Layers shown in Fig. 1 where there is an intense agitation.

In a laminar boundary layer any exchange of mass or momentum takes place only between adjacent layers on a microscopic scale which is not visible to the eye. Consequently molecular viscosity ${ }^{\mu}$ is able predict the shear stress associated. Laminar boundary layers are found only when the Reynolds numbers are small.



Figure 1: Typical velocity profiles for laminar and turbulent boundary layers

A turbulent boundary layer on the other hand is marked by mixing across several layers of it. The mixing is now on a macroscopic scale. Packets of fluid may be seen moving across. Thus there is an exchange of mass, momentum and energy on a much bigger scale compared to a laminar boundary layer. A turbulent boundary layer forms only at larger Reynolds numbers. The scale of
mixing cannot be handled by molecular viscosity alone. Those calculating turbulent flow rely on what is called Turbulence Viscosity or Eddy Viscosity, which has no exact expression. It has to be modelled. Several models have been developed for the purpose.


Figure 2: Typical velocity profiles for laminar and turbulent boundary layers

## Dimensional Analysis for Free and Forced Convection:

If number of variables influencing convective heat transfer coefficient are known, then the following two methods can be used to develop a mathematical expression relating the variables with the convective heat transfer coefficient.
i) Rayleigh's Method
ii) Buckingham's $\pi$-theorem

However, in application of dimensional analysis for determining convective heat transfer coefficient for free and forced convection, Rayleigh's method will not be used as it has certain limitations that can be overcome by using Buckingham's $\pi$-theorem method.

## Buckingham's $\boldsymbol{\pi}$-Theorem Method

In the Rayleigh's method of dimensional analysis, solution becomes more and more cumbersome and laborious if number of influencing variables become more than the fundamental units ( $\mathrm{M}, \mathrm{L}, \mathrm{T}$ and $\theta$ ) involved in the physical phenomenon.. The use of Buckingham's $\pi$-theorem method enables to overcome this limitation and states that if there are ' $n$ ' variables (independent and dependent) in a physical phenomenon and if these variables contain ' $m$ ' number of fundamental dimensions ( $\mathrm{M}, \mathrm{L}, \mathrm{T}$ and $\theta$ ), then the variables are arranged in to ( $\mathrm{n}-\mathrm{m}$ ) dimensionless terms called $\pi$-terms.

Buckingham's $\pi$-Theorem Method can be applied for forced and free convection processes to determine the heat transfer coefficient.

## Dimensional Analysis for Forced Convection

On the basis of experience, it is concluded that forced convection heat transfer coefficient is a function of variables given below in Table -2

| S. No. | Variable / Parameter | Symbol | Dimensions |
| :--- | :--- | :--- | :--- |
| 1 | Fluid density | $\rho$ | $\mathrm{ML}^{-3}$ |
| 2 | Dynamic viscosity of fluid | $\mu$ | $\mathrm{ML}^{-1} \mathrm{~T}^{-1}$ |
| 3 | Fluid Velocity | V | $\mathrm{LT}^{-1}$ |
| 4 | Thermal conductivity of fluid | k | $\mathrm{MLT}^{-3} \theta^{-1}$ |
| 5 | Specific heat of fluid | $\mathrm{C}_{\mathrm{p}}$ | $\mathrm{L}^{2} \mathrm{~T}^{-2} \theta^{-1}$ |


| 6 | Characteristic length of heat transfer area | D | L |
| :--- | :--- | :--- | :--- |

Therefore, convective heat transfer coefficient is expressed as
$\mathrm{h}=\mathrm{f}\left(\rho, \mu, \mathrm{V}, \mathrm{k}, \mathrm{C}_{\mathrm{p}}, \mathrm{D}\right)$
$\mathrm{f}\left(\mathrm{h}, \rho, \mu, \mathrm{V}, \mathrm{k}, \mathrm{C}_{\mathrm{p}}, \mathrm{D}\right)=0$
Convective heat transfer coefficient, h is dependent variable and remaining are independent variables.
Total number of variables, $\mathrm{n}=7$
Number of fundamental units, $\mathrm{m}=4$
According to Buckingham's $\pi$-theorem, number of $\pi$-terms is given by the difference of total number of variables and number of fundamental units.

Number of $\pi$-terms $=(\mathrm{n}-\mathrm{m})=7-4=3$
These non-dimensional $\pi$-terms control the forced convection phenomenon and are expressed as
$\mathrm{f}\left(\pi_{1}, \pi_{2}, \pi_{3}\right)=0$
Each $\pi$-term is written in terms of repeating variables and one other variable. In order to select repeating variables following method should be followed.

- Number of repeating variables should be equal to number of fundamental units involved in the physical phenomenon.
- Dependent variable should not be selected as repeating variable.
- The repeating variables should be selected in such a way that one of the variables should contain a geometric property such as length, diameter or height. Other repeating variable should contain a flow property such as velocity or acceleration and the third one should contain a fluid property such as viscosity, density, specific heat or specific weight.
- The selected repeating variables should not form a dimensionless group.
- The selected repeating variables together must have same number of fundamental dimensions.
- No two selected repeating variables should have same dimensions.

The following repeating variables are selected
i) Dynamic viscosity, $\mu$ having fundamental dimensions $\mathbf{M L}^{-1} \mathbf{T}^{-1}$
ii) Thermal conductivity, k having fundamental dimensions $\mathrm{MLT}^{-3} \boldsymbol{\theta}^{-1}$
iii) Fluid velocity, V having fundamental dimensions $\mathrm{LT}^{-1}$
iv) Characteristic length, $\mathbf{D}$ having fundamental dimensions $L$

Each $\pi$-term is expressed as:
$\pi_{1}=\mu^{\mathrm{a}} \mathrm{k}^{\mathrm{b}}, \mathrm{V}^{\mathrm{c}}, \mathrm{D}^{\mathrm{d}}, \mathrm{h}$
Writing down each term in above equation in terms of fundamental dimensions
$\mathrm{M}^{0} \mathrm{~L}^{0} \mathrm{~T}^{0} \theta^{0}=\left(\mathrm{ML}^{-1} \mathrm{~T}^{-1}\right)^{\mathrm{a}}\left(\mathrm{MLT}^{-3} \theta^{-1}\right)^{\mathrm{b}}\left(\mathrm{LT}^{-1}\right)^{\mathrm{c}}(\mathrm{L})^{\mathrm{d}} \mathrm{MT}^{-3} \theta^{-1}$
Comparing the powers of M , we get
$0 \quad=\quad \mathrm{a}+\mathrm{b}+1, \quad \mathrm{a}+\mathrm{b}=\quad-1$ (5)

Comparing powers of $L$, we get
$0=-a+b+c+d$
Comparing powers of T , we get

$$
\begin{equation*}
0=-a-3 b-c-3 \tag{7}
\end{equation*}
$$

Comparing powers of $\theta$, we get
$0=-\mathrm{b}-1$,
$\mathrm{b}=-1$
Substituting value of 'b' from equation (8) in equation (5), we get $\mathrm{a}=0$

Substituting values of ' $a$ ' and ' $b$ ' in equation (7), we get
$\mathrm{c}=0$
Substituting the values of ' $a$ ', ' $b$ ' and ' $c$ ' in equation (6), we get
$\mathrm{d}=1$
Substituting the values of ' $a$ ', ' $b$ ', ' $c$ ' and ' $d$ ' in equation (4), we get
$\pi_{1}=\mu^{-0} \mathrm{k}^{-1}, \mathrm{~V}^{0}, \mathrm{D}^{1}, \mathrm{~h}$
$\pi_{1}=\mathrm{hD} / \mathrm{k}$
The second $\pi$-term is expressed as
$\pi_{2}=\mu^{\mathrm{a}} \mathrm{k}^{\mathrm{b}}, \mathrm{V}^{\mathrm{c}}, \mathrm{D}^{\mathrm{d}}, \rho$
$\mathrm{M}^{0} \mathrm{~L}^{0} \mathrm{~T}^{0} \theta^{0}=\left(\mathrm{ML}^{-1} \mathrm{~T}^{-1}\right)^{\mathrm{a}}\left(\mathrm{MLT}^{-3} \theta^{-1}\right)^{\mathrm{b}}\left(\mathrm{LT}^{-1}\right)^{\mathrm{c}}(\mathrm{L})^{\mathrm{d}} \mathrm{ML}^{-3}$
Comparing the powers of M , we get
$0 \quad=\quad a+b+1, \quad a+b=-1$

Comparing powers of L , we get
$0=-a+b+c+d-3$
Comparing powers of T , we get
$0=-a-3 b-c$
Comparing powers of $\theta$, we get
$0=-b$,
$\mathrm{b}=0$
Substituting value of ' $b$ ' from equation (16) in equation (13), we get
$a=-1$

Substituting values of ' $a$ ' and ' $b$ ' in equation (15), we get
$\mathrm{c}=1$

Substituting the values of 'a', ' $b$ ' and ' $c$ ' in equation (14), we get
$d=1$

Substituting the values of 'a', 'b', 'c' and 'd' in equation (12), we get
$\pi_{2}=\mu^{-1} \mathrm{k}^{0}, \mathrm{~V}^{1}, \mathrm{D}^{1}, \rho$
$\pi_{2}=\rho \mathrm{VD} / \mu$
The third $\pi$-term is expressed as
$\pi_{3}=\mu^{\mathrm{a}} \mathrm{k}^{\mathrm{b}}, \mathrm{V}^{\mathrm{c}}, \mathrm{D}^{\mathrm{d}}, \mathrm{C}_{\mathrm{p}}$
$\mathrm{M}^{0} \mathrm{~L}^{0} \mathrm{~T}^{0} \theta^{0}=\left(\mathrm{ML}^{-1} \mathrm{~T}^{-1}\right)^{\mathrm{a}}\left(\mathrm{MLT}^{-3} \theta^{-1}\right)^{\mathrm{b}}\left(\mathrm{LT}^{-1}\right)^{\mathrm{c}}(\mathrm{L})^{\mathrm{d}} \mathrm{L}^{2} \mathrm{~T}^{-2} \theta^{-1}$
Comparing the powers of M , we get
$0=a+b, a+b=0$
Comparing powers of L , we get
$0=-a+b+c+d+2$

Comparing powers of T , we get

$$
\begin{equation*}
0=-\mathrm{a}-3 \mathrm{~b}-\mathrm{c}-2 \tag{23}
\end{equation*}
$$

Comparing powers of $\theta$, we get
$0=-b-1$,
$\mathrm{b}=-1$
Substituting value of ' $b$ ' from equation (24) in equation (21), we get
$\mathrm{a}=1$
Substituting values of ' $a$ ' and ' $b$ ' in equation (23), we get
$\mathrm{c}=0$
Substituting the values of ' $a$ ', ' $b$ ' and ' $c$ ' in equation (22), we get
$\mathrm{d}=0$
Substituting the values of ' $a$ ', ' $b$ ', ' $c$ ' and ' $d$ ' in equation (12), we get
$\pi_{3}=\mu^{1} \mathrm{k}^{-1}, \mathrm{~V}^{0}, \mathrm{D}^{0}, \mathrm{C}_{\mathrm{p}}$
$\pi_{3}=\mu \mathrm{C}_{\mathrm{p}} / \mathrm{k}$
Substituting the values of $\pi_{1}, \pi_{2}, \pi_{3}$ in equation (3), we get
$\mathrm{f}\left(\mathrm{h} \mathrm{D} / \mathrm{k}, \rho \mathrm{VD} / \mu, \mu \mathrm{C}_{\mathrm{p}} / \mathrm{k}\right)=0$
$\mathrm{h} \mathrm{D} / \mathrm{k}=\varphi\left(\rho \mathrm{VD} / \mu, \mu \mathrm{C}_{\mathrm{p}} / \mathrm{k}\right)$
$\mathrm{Nu}=\varphi(\operatorname{Re}, \mathrm{Pr})$
The above correlation is generally expressed as
$\mathrm{Nu}=\mathrm{C}(\operatorname{Re})^{\mathrm{a}}(\operatorname{Pr})^{\mathrm{b}}$
The constant C and exponents ' a ' and ' $b$ ' are determined through experiments.

## Dimensional Analysis for Free Convection:

In free convection heat transfer process, convective heat transfer coefficient depends upon the same parameters/variable as in case of forced convection except velocity of fluid. It is on account of the fact that in free convection motion of fluid occurs due to difference in density of various layers of fluid caused by temperature difference whereas in case of forced convection motion of fluid is caused by an external source. The fluid velocity in case of free convection depends upon the following parameters;

## i) Temperature difference between solid surface and bulk fluid, $\Delta T$

## ii) Acceleration due to gravity, g

iii) Coefficient of volumetric expansion of fluid, $\beta$

The change in the volume when temperature changes can be expressed as
$\mathrm{dV}=\mathrm{V}_{1} \beta\left(\mathrm{~T}_{2}-\mathrm{T}_{1}\right)$
where
$d V$ - change in volume $\left(\mathrm{m}^{3}\right)$

$$
=\mathrm{V}_{2}-\mathrm{V}_{1}
$$

$\beta=$ Coefficient of volumetric expansion of fluid, $\left(\mathrm{m}^{3} / \mathrm{m}^{3}{ }^{\circ} \mathrm{C}\right)$
$\mathrm{T}_{2}$ - Final temperature ( ${ }^{\circ} \mathrm{C}$ )
$\mathrm{T}_{1}$ - Initial temperature ( ${ }^{\circ} \mathrm{C}$ )
Therefore, free convection heat transfer coefficient is a function of variables given in Table 3
Table 3

| S. No. | Variable | Symbol | Dimensions |
| :--- | :--- | :--- | :--- |
| 1 | Fluid density | $\rho$ | $\mathrm{ML}^{-3}$ |
| 2 | Dynamic viscosity of fluid | $\mu$ | $\mathrm{ML}^{-1} \mathrm{~T}^{-1}$ |
| 3 | Thermal conductivity of fluid | k | $\mathrm{MLT}^{-3} \theta^{-1}$ |
| 4 | Specific heat of fluid | D | $\mathrm{L}^{2} \mathrm{~T}^{-2} \theta^{-1}$ |
| 5 | Characteristic length of heat transfer area | $\beta$ | $\theta^{-1}$ |
| 6 | Temperature difference between surface and bulk fluid | $\Delta \mathrm{T}$ | $\theta$ |
| 7 | Coefficient of volumetric expansion | g | $\mathrm{LT}^{-2}$ |
| 8 | Acceleration due to gravity | $\beta$ |  |

Therefore, convective heat transfer coefficient is expressed as
$h=f\left(\rho, \mu, k, C_{p}, D, \Delta T, \beta, g\right)$
However, in free convection, ( $\Delta \mathrm{T} \beta \mathrm{g}$ ) will be treated as single parameter as the velocity of fluid particles is a function of these parameters. Therefore, equation (28) can be expressed as
$\mathrm{f}\left(\mathrm{h}, \rho, \mu, \mathrm{k}, \mathrm{C}_{\mathrm{p}}, \mathrm{D},(\Delta \mathrm{T} \beta \mathrm{g})\right)=0$
Convective heat transfer coefficient, h is dependent variable and remaining are independent variables.
Total number of variables, $n=7$
Number of fundamental units, $\mathrm{m}=4$
According to Buckingham's $\pi$-theorem, number of $\pi$-terms is given by the difference of total number of variables and number of fundamental units.

Number of $\pi$-terms $=(n-\mathrm{m})=7-4=3$
These non-dimensional $\pi$-terms control the forced convection phenomenon and are expressed as
$\mathrm{f}\left(\pi_{1}, \pi_{2}, \pi_{3}\right)=0$
Each $\pi$-term is written in terms of repeating variables and one other variable and the following repeating variables are selected
i) Dynamic viscosity, $\mu$ having fundamental dimensions $\mathrm{ML}^{-1} \mathrm{~T}^{-1}$
ii) Thermal conductivity, k having fundamental dimensions MLT $^{-3} \boldsymbol{\theta}^{-1}$
iii) Fluid density, $\rho$ having fundamental dimensions $\mathrm{ML}^{-3}$

## iv) Characteristic length, $\mathbf{D}$ having fundamental dimensions $L$

Each $\pi$-term is expressed as:
$\pi_{1}=\mu^{\mathrm{a}} \mathrm{k}^{\mathrm{b}}, \rho^{\mathrm{c}}, \mathrm{D}^{\mathrm{d}}, \mathrm{h}$
Writing down each term in above equation in terms of fundamental dimensions
$\mathrm{M}^{0} \mathrm{~L}^{0} \mathrm{~T}^{0} \theta^{0}=\left(\mathrm{ML}^{-1} \mathrm{~T}^{-1}\right)^{\mathrm{a}}\left(\mathrm{MLT}^{-3} \theta^{-1}\right)^{\mathrm{b}}\left(\mathrm{ML}^{-3}\right)^{\mathrm{c}}(\mathrm{L})^{\mathrm{d}} \mathrm{MT}^{-3} \theta^{-1}$
Comparing the powers of M , we get
$0 \quad=\quad a+b+c+1, \quad a+b+c=\quad-1$
(32)

Comparing powers of L , we get
$0=-a+b+c+d$

Comparing powers of T , we get
$0=-\mathrm{a}-3 \mathrm{~b}-\mathrm{c}-3$
Comparing powers of $\theta$, we get
$0=-\mathrm{b}-1$,
$b=-1$
Substituting value of ' $b$ ' from equation (35) in equation (32), we get
$\mathrm{a}=0$
Substituting values of ' $a$ ' and ' $b$ ' in equation (34), we get
$\mathrm{c}=0$
Substituting the values of ' $a$ ', ' $b$ ' and ' $c$ ' in equation (33), we get
$\mathrm{d}=1$
Substituting the values of ' $a$ ', 'b', 'c' and ' $d$ ' in equation (31), we get
$\pi_{1}=\mu^{\mathrm{a}} \mathrm{k}^{\mathrm{b}}, \rho^{\mathrm{c}}, \mathrm{D}^{\mathrm{d}}, \rho$
$\pi_{1}=\mathrm{hD} / \mathrm{k}$
The second $\pi$-term is expressed as
$\pi_{2}=\mu^{\mathrm{a}} \mathrm{k}^{\mathrm{b}}, \rho^{\mathrm{c}}, \mathrm{D}^{\mathrm{d}}, \mathrm{C}_{\mathrm{p}}$
$\mathrm{M}^{0} \mathrm{~L}^{0} \mathrm{~T}^{0} \theta^{0}=\left(\mathrm{ML}^{-1} \mathrm{~T}^{-1}\right)^{\mathrm{a}}\left(\mathrm{MLT}^{-3} \theta^{-1}\right)^{\mathrm{b}}\left(\mathrm{ML}^{-3}\right)^{\mathrm{c}}(\mathrm{L})^{\mathrm{d}} \mathrm{L}^{2} \mathrm{~T}^{-2} \theta^{-1}$
Comparing the powers of M , we get

Comparing powers of L , we get
$0=-a+b-3 c+d+2$
Comparing powers of T , we get
$0=-a-3 b-2$
Comparing powers of $\theta$, we get
$0 \quad=\quad$ (43) $\quad-b-1, \quad b=-1$

Substituting value of 'b' from equation (43) in equation (40), we get
$\mathrm{a}=1$
Substituting values of ' $a$ ' and ' $b$ ' in equation (42), we get
$\mathrm{c}=0$
Substituting the values of ' $a$ ', ' $b$ ' and ' $c$ ' in equation (41), we get
$\mathrm{d}=0$
Substituting the values of ' $a$ ', ' $b$ ', ' $c$ ' and ' $d$ ' in equation (39), we get
$\pi_{2}=\mu^{1} \mathrm{k}^{-1}, \rho^{0}, \mathrm{D}^{0}, \mathrm{C}_{\mathrm{p}}$
$\pi_{2}=\mu \mathrm{C}_{\mathrm{p}} / \mathrm{k}=$ Prandtl Number $=\operatorname{Pr}$

The third $\pi$-term is expressed as
$\pi_{3}=\mu^{\mathrm{a}} \mathrm{k}^{\mathrm{b}}, \rho^{\mathrm{c}}, \mathrm{D}^{\mathrm{d}},((\Delta \mathrm{T} \beta \mathrm{g})$
$\mathrm{M}^{0} \mathrm{~L}^{0} \mathrm{~T}^{0} \theta^{0}=\left(\mathrm{ML}^{-1} \mathrm{~T}^{-1}\right)^{\mathrm{a}}\left(\mathrm{MLT}^{-3} \theta^{-1}\right)^{\mathrm{b}}\left(\mathrm{ML}^{-3}\right)^{\mathrm{c}}(\mathrm{L})^{\mathrm{d}}\left(\theta^{-1} \mathrm{LT}^{-2} \theta^{1}\right)$
$\mathrm{M}^{0} \mathrm{~L}^{0} \mathrm{~T}^{0} \theta^{0}=\left(\mathrm{ML}^{-1} \mathrm{~T}^{-1}\right)^{\mathrm{a}}\left(\mathrm{MLT}^{-3} \theta^{-1}\right)^{\mathrm{b}}\left(\mathrm{ML}^{-3}\right)^{\mathrm{c}}(\mathrm{L})^{\mathrm{d}}\left(\mathrm{LT}^{-2}\right)$
Comparing the powers of M , we get
$0 \quad=\quad a+b+c, \quad a+b+c=\quad 0$

Comparing powers of L , we get
$0=-a+b-3 c+d+1$
Comparing powers of T , we get
$0=-a-3 b-2$
Comparing powers of $\theta$, we get
$0=-b, b=0$
Substituting value of 'b' from equation (51) in equation (48), we get

$$
\begin{equation*}
a=-2 \tag{52}
\end{equation*}
$$

Substituting values of 'a' and 'b' in equation (50), we get
$\mathrm{c}=2$

Substituting the values of 'a', 'b' and 'c' in equation (49), we get
$d=3$

Substituting the values of 'a', 'b', 'c' and 'd' in equation (47), we get
$\pi_{3}=\mu^{-2} \mathrm{k}^{0}, \rho^{2}, \mathrm{D}^{3},(\Delta \mathrm{~T} \beta \mathrm{~g})$
$\pi_{3}=\rho^{2} D^{3}(\Delta T \beta g) / \mu^{2}$
$=$
$D^{3}(\Delta T$
$v^{2}$
$\beta$
(54)

Substituting the values of $\pi_{1}, \pi_{2}, \pi_{3}$ in equation (30), we get
$\mathrm{f}\left(\mathrm{h} \mathrm{D} / \mathrm{k}, \mu \mathrm{C}_{\mathrm{p}} / \mathrm{k}, \mathrm{D}^{3}(\Delta \mathrm{~T} \beta \mathrm{~g}) / \mathrm{v}^{2}\right)=0$
$\mathrm{h} D / \mathrm{k}=\varphi\left(\mu \mathrm{C}_{\mathrm{p}} / \mathrm{k}, \mathrm{D}^{3}(\Delta \mathrm{~T} \beta \mathrm{~g}) / \mathrm{v}^{2}\right)$
$\mathrm{Nu}=\varphi(\operatorname{Pr}, \mathrm{Gr}) \quad$ as $\mathrm{Gr}=\mathrm{D}^{3}(\Delta \mathrm{~T} \beta \mathrm{~g}) / v^{2}$
The above correlation is generally expressed as
$\mathrm{Nu}=\mathrm{C}(\operatorname{Pr})^{\mathrm{a}}(\mathrm{Gr})^{\mathrm{b}}$
The constant C and exponents a and b are determined through experiments.

